THE DYNAMICS

OF

NATIONAL ECONOMIC SYSTEMS

by

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STATEMENT

The work presented in this thesis is original with the exceptions stated below, and has not been submitted for another degree of the University of Warwick or any other Institution. The exceptions are:

(i) Throughout the emphasis has been one of systematically applying known mathematical results and systems techniques to solve the problems at hand. Many of these methods are well documented in engineering literature, and it is only aspects of their application to economic systems that is new.

(ii) The actual systems defined in chapters 2-4 have been derived from the literature cited. In particular the linear system defined by model IA was proposed by Professors F. G. Pyatt and P. C. Parks, and has also been analysed by A. Hulme as is indicated in the references.

(iii) The concept of models which can generate cycles through unemployment disequilibrium dynamics is built up from model II, and was due initially to Professor R. M. Goodwin. The solution of model II in section 3.1 is not original, but is included for the sake of completeness.

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ABSTRACT

This thesis examines the behaviour of a number of macro-economic models starting initially with a simple linear formulation which is then developed systematically to embrace an increasing degree of economic sophistication and consequent mathematical complexity. The study is essentially an exercise in mathematical economics as opposed to econometrics, (the object being to explain behavioural characteristics rather than to extrapolate forecasts) and the techniques employed are those of *systems dynamics*.

Following the introductory remarks and definitions of Chapter 1, Chapter 2 deals exclusively with continuous linear models and their solutions, which permit a number of limited conclusions to be drawn concerning the rates of growth of certain key variables such as investment and output. However, in the presence of disturbances, exponential growth is merely a trend about which fluctuations will occur on account of the dynamic adjustment mechanisms which then come into play. Two such mechanisms are considered in the context of a number of further models in Chapter 3, which are designed in such a way as to permit at least approximate analytic solutions.

From this point in the thesis more detailed economic considerations (in particular the concept of a vintage technology) lead to the formulation of differential-delay equation models of a class which has hitherto received little attention in the literature. The basic models of this type are formulated in Chapter 4 which also examines the existence and uniqueness of *equilibrium growth* solutions.

Having determined such solutions, Chapter 5 is concerned with the application of a number of standard control theory techniques as a means of establishing local stability conditions for equilibrium growth.

Chapter 6 is concerned generally with the digital computer solution of differential-delay equations and specifically with the simulation of an advanced national economy as represented by such a system of equations. A number of interesting numerical problems arise in the simulation for which data relating to the United Kingdom is used. Some results are presented and a full documentation of the simulation programs is given in the appendix.

ACKNOWLEDGEMENTS

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LIST OF SYMBOLS

Variables

C consumption

D dividends

X exports

G government (public) expenditure

I gross national investment

I_f foreign investment

K total stock of capital

E national labour employment

L labour supply (= $L_0\ exp\{\beta t\}$ where L_0 constant)

M imports

$N(\tau)$ labour employment on plant of vintage τ

P price index

R profits

V present value of an investment

S savings

S_c company savings

S_g government savings

S_h household savings

T taxes

T_c company taxes

T_h household taxes

W national wage bill

$Z(\tau)$ output from plant of vintage τ

Y gross national product

U unemployment

c	$= k\ exp\{-\alpha t\}$	
k	$= I/N$, per capita investment	
n	$= N/L$	
r	rate of interest or rate of return on an investment	
u	$= W/Y$, share of wages	
v	$= E/L$, relative employment	
w	$= W/E$, per capita wage rate	
y	$= Y/E$, per capita output	
a	productivity of labour	
θ	economic life of plant or age of oldest plant	
ϕ	anticipated economic life of new plant	
λ	anticipated growth rate of wages relative to prices	
μ	anticipated growth rate of prices	

Parameters

α	growth rate of technical progress
β	growth rate of labour supply
g_C	growth rate of consumption
g_K	growth rate of capital stock
g_X	growth rate of exports
s_w	savings rate of wages
s_p	savings rate on profits
t_h	tax rate on private incomes
t_c	tax rate on company profits
t_a	rate of tax allowance on investments
σ	capital-output ratio
γ, ρ, η	wage equation parameters

ξ price equation parameter

b Cobb-Douglas production function parameter

CHAPTER 1

INTRODUCTION

The aim of this thesis is to demonstrate the application of control techniques and simulation in the process of building and analysing mathematical models of a national economy, and hopefully through the models which are developed to obtain results which are both meaningful and relevant in terms of macroeconomics.

A mathematical model of an economic system may be defined as *a specification in mathematical terms of the inter-relationships of the system, constructed for the purpose of controlling or predicting the future behaviour of the system.* As such the process of building a mathematical model of an economic system is no different from that applied in constructing a model of any other social or physical system and indeed the eventual mathematical specifications of two models, each of entirely different situations, might be identical; to take an example which arises later in this text the same model may represent either investment growth in a national economy or the spread of epidemics in a human community. However, the manner in which a model is built depends upon the purpose for which it is designed, for the aim in building a mathematical model is not to provide an accurate description of the system to be used for all possible purposes. Thus one may with equal validity construct different models of the same system. In economic modelling particularly the aim is not to describe reality but rather to feature only those characteristics which have a bearing on the decisions under investigation, and thereby to reduce the complexities of the real system to a more manageable form. The process is one which frequently commences with a relatively modest model, perhaps one for which an analytic solution can be found. It progresses, often by some form of trial and error, by introducing additional structures and variables whose relevance is then tested against the particular purpose of the modelling exercise. On the one hand a national economic model might be used by a firm simply to predict the time path of certain variables in the short to medium term (0 → 5 years); or alternatively a model of the same economy may be required by the government for actual control and policy decision making.

Of course the very nature of subject makes modelling in economics an extremely difficult task when for instance the complex decisions which are implemented by corporations may have to be predicted and understood. Even the measurement of variables, which probably presents little difficulty in a physical system, entails conversion to a money value where the conversion mechanism itself is part of the dynamics of the system. Then there is the problem of aggregation - to what extent can the behaviour of an average (man, price, wage, etc.) represent the collective behaviour of the separate individual components? It may be that the concept of aggregation is a perfectly valid divergence from reality for many exercises, but the question nevertheless remains to be satisfactorily answered.

Under these generally adverse circumstances it is hardly surprising that the *state of the art* in economic modelling and forecasting is not advanced; but it is surely certain that further significant progress will come about only by the utilisation of every available tool of which control theory is just one. The application of control techniques to economic concepts dates back to the now well known and very remarkable pioneering work of Tustin[63] which was published in 1953. Shortly afterwards Phillips[53, 54] (again an engineer by initial training) also published his work formulating economic models in the terminology of the systems engineer and applying established control methods relating to stabilisation. It is however perhaps surprising that this early work did not produce much follow-up until fairly recently when activity was again revived by such workers as Dobell[17], Livesey[42] and Noton[47] among others.

It has been suggested by Parks and Pyatt[51] that reasons for this latest resurgence of interest are the rapid progress in control techniques in the past 20 years on the one hand, and the advancement of economics in its ability to formulate mathematical models and in the estimation of their parameters on the other. In economic growth in particular the number of important contributions to modern theory have been tremendous (see the surveys and collected papers edited by Hahn[25, 26]). These have examined neo-classical growth models of economies with fixed coefficients and vintage models with a particular emphasis on the existence, uniqueness and efficiency of equilibrium or steady state solutions. On the other hand the theory of optimal control has grown and flourished in the same period and it is notable that most of the recent interdisciplinary work referred to above has been concerned with the application of optimal control.

The use of optimal control theory in economics has itself served to highlight another issue which concerns the goals of economic policy, or in control terminology, the definition of suitable *objective* or *welfare* functions. It seems to be generally agreed that the sustainment of economic growth should be a primary objective but there are many other issues which in any instance will be in need of resolution, such as the relative weight or *discount* to be given to consumption flows in the distant as opposed to the more immediate future, the short term problems of unemployment and balance of payments (foreign trade account) control, the distribution of wealth and so on. With the need for much further work in this area alone it may be that the application of optimal control theory to economics, beyond what has already been demonstrated, would at this stage be premature. It is also important at this point to appreciate the heavy reliance placed upon high speed digital computers by most workers when solving their optimisation algorithms, which is a further reason why these developments could not have taken place earlier.

It is the advent of the modern digital computer in its present form which has given impetus to another recent trend in economic model building, namely the construction and simulation of large scale *econometric* models used primarily for short term forecasting. Simulation in this context is a numerical technique for conducting experiments on a digital computer which involves first setting up a mathematical model of the economic system on the computer and subsequently deriving forecasts from the model. Important early contributions in this field of activity were the Klein-Goldberger models[22] of the United States economy and the United Kingdom model[65]. Subsequently models of this type have been developed with as many as 1500 equations including a model of the U.S. economy at the Wharton Business School and in the U.K., the London Graduate Business School model, the Treasury model and the National Institute of Social and Economic Research model. These large *data-based* models have been built to meet the needs of government and industry for detailed short term forecasts, but they are subject to the criticism that their mechanisms are not fully understood by the model authors themselves. However even with these models work is now being done using the frequency response tests and pole and zero searches familiar to the control engineer, in addition to which it can justifiably be claimed that continual daily experimentation with a model by digital simulation does lead to a strong *feel* for its behavioural mechanism. Computer simulation is thus an extremely important tool in the hands of the economic analyst.

The *data-based* model has lead to a great deal of research in parameter estimation by such workers as Theil[62]. Generally a problem arises due to the extensive *coupling* between economic variables - in a simple but classic example we know that wage increases give rise to price increases but that the latter also give rise to the former - and special econometric techniques in multi-stage parameter estimation are needed in such circumstances. Even when these difficulties have been overcome, with large multi-sector models it is not possible to forecast beyond the short term because of the build-up of large error variance. Forecasts made by the British Treasury model are limited to a time horizon of 2 years, which confinement is clearly less than adequate for government policy determination in such a field as investment for example. Thus we are forced to reconsider models of a *conceptual* nature which are *non data-based* (and thereby immediately vulnerable to the criticism that they are unrelated to reality). This is the distinction between the econometric model and that of mathematical economics; but a role for each is quite clear. (Even Forrester's renowned world model[21] and the work which it preceeded is an example of a *conceptual* model which many would argue has fulfilled a valid role.)

It is to the category of mathematical economics that the present thesis aims to contribute in the belief that here perhaps is potentially the most significant area for control theory applications. The philosophy has been to present a sequence of *time variable* or *dynamic* models (all of a *conceptual* nature) and to analyse these models using appropriate techniques in control theory and simulation. The term *systems dynamics* is commonly used to describe this field of activity. *Systems dynamics* unlike econometrics does not lean heavily upon numbers until the latter stages of an investigation and its use is concerned with a greater emphasis on explanatory reasons. It accepts as an operating principle that no complex system can be fully known in all its interactive detail, and accordingly seeks to elucidate global properties that characterise *core* dynamics.

The first models are concerned solely with steady state growth and their analysis generates a number of elementary results relating the growth rates of various sectors of the economy (and one may recall that the growth rate of government expenditure was identified by Keynes in his *General Theory*[37] as a possible control to counter economic depression). Subsequent models are concerned with the issue of cyclical gorwth. The essential question to which this thesis is addressed is what is the effect when short/medium term disequilibrium adjustment mechanisms are superimposed on the basic model of long term growth. Thus we are led to discuss the stability of steady state growth and to examine the possibility of persistent oscillations as may be relevant to theories concerning the *business cycle* (see Matthews[43] and the work of Goodwin[23], Harrod[29], Hicks[30] and Hansen[28]). All models used are expressed in terms of continuous as opposed to discrete time not because the former is regarded as inherently superior but merely motivated by mathematical convenience. In fact in terms of the digital simulation which was used to analyse most of the models the choice of a discrete time formulation would have avoided a number of serious difficulties which are discussed at the appropriate point in the text. For each simulation of the more complex models a program listing is given not only so that the results may be reproduced or extended, but also because this is an ideal means of specifying each model in a way which is absolutely unambiguous.

The construction of any macroeconomic model may be generalized in terms of three phases. These are

(a) definition of the variables,

(b) formulation of identity relationships that are self-evident following the manner in which the variables have been defined, and

(c) specification of a set of behavioural relationships as is necessary to complete the model.

Variables are of either an exogenous or endogenous nature - thus for example labour supply in the present work is of the former category; that is to say that it is determined by factors completely outside the scope of the model although it may still be subject to variation so that typically in the case of labour supply, for example, one might assume exponential growth. As a further example it might be assumed in a national model that the price of raw material imports is exogenous. All other variables are defined as endogenous or determined by the internal system mechanisms. Thus will be the case with the national output determined perhaps partly by national demand or more directly by such factors as labour input, personal incomes, past consumption, profits and dividends, savings and investment, and by capacity in terms of total capital stock.

With the single exception of capital stock each of the variables mentioned above represents a money flow. Fig. 1 (a) shows the flow of goods and the provision of services in the simple case of a closed economy with only 3 markets - for goods, labour services and capital services. The household block provides production activities with labour services and in return receives consumer goods. Production activities also make use of capital services and supply investment goods. Fig. 1 (b) shows the flow of monies in the reverse direction between the household, production and capital accounts. It is evident that by the principles of accounting these money flows must balance to give three identity relationships (the so called accounting identities) which are as follows:

PRODUCTION ACCOUNT

Consumer Expenditure + Investment Expenditure

≡ Wages + Profits *(1.1)*

HOUSEHOLD ACCOUNT

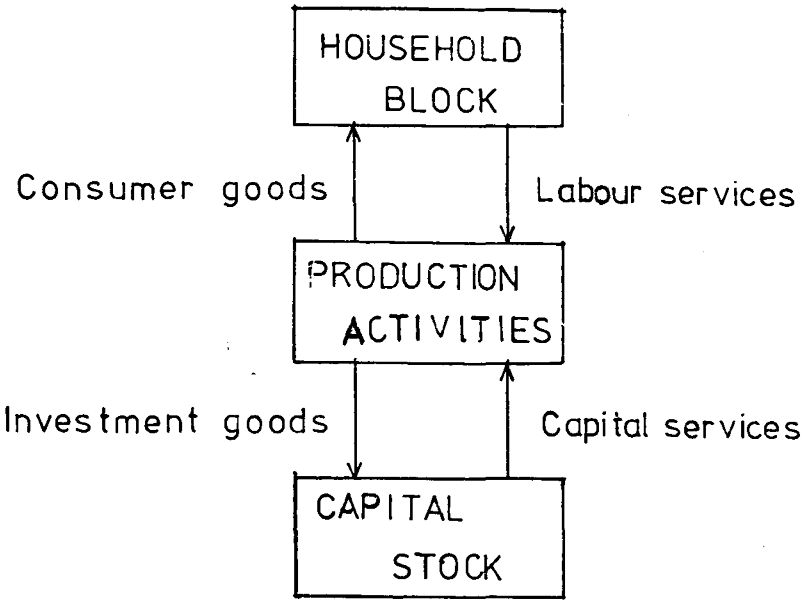
Wages + Dividends ≡ Consumer Expenditure + Savings *(1.2)*

CAPITAL ACCOUNT

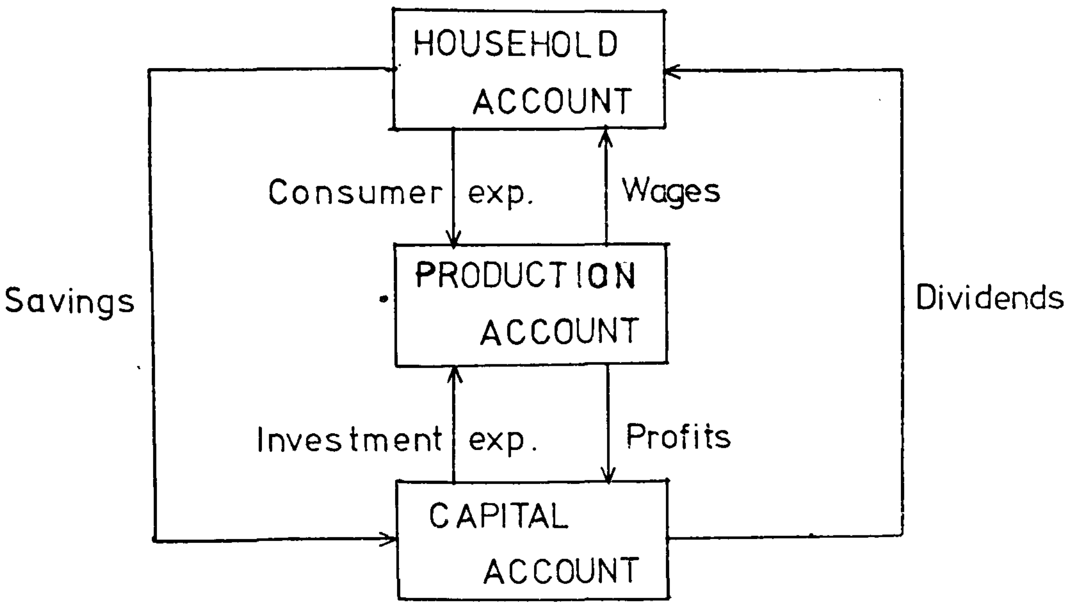
Profits + Savings ≡ Investment Expenditure + Dividends *(1.3)*

In addition to these identity relationships is the relationship between capital stock and investment expressed as

Investment Flow = Gross Rate of Increase of Capital Stock *(1.4)*



(a) Flow of goods and services



(b) Flow of money

FIGURE 1 Block diagram of a simple economy

Lastly in the modelling exercise behavioural relationships are formulated describing the operation of the various sectors - a simple example might be to postulate the manner in which households determine the disposal of their income. For a properly determined model the total number of identity relationships plus the number of behavioural relationships must equal the number of endogenous variables and the model is then solvable.

In the following chapter discussion is limited to behavioural relationships which are linear.

CHAPTER 2

LINEAR MODELS

The general linear differential model with constant coefficients is described in vector-matrix form by the equations

$$\left.\begin{aligned} \frac{d\underline{x}}{dt} &= A_1\underline{x} + B_1\underline{y} + C_1\underline{z} \\ 0 &= A_2\underline{x} + B_2\underline{y} + C_2\underline{z} \end{aligned}\right\} \qquad (2.1)$$

where the vectors $\underline{x} = [x_1, x_2, \ldots, x_k]'$ and $\underline{y} = [y_1, y_2, \ldots, y_m]'$ together represent the set of $n = k + m$ time dependent endogenous variables and $\underline{z} = [z_1, z_2, \ldots, z_r]'$ is the set of exogenous variables. Each of the matrices A_i, B_i and C_i $(i = 1, 2)$ are constant and for a properly determined system A_1 is of order $k \times k$, A_2 is of order $m \times k$, B_1 is of order $k \times m$, B_2 is of order $m \times m$ and is non-singular, C_1 is of order $k \times r$ and C_2 is of order $m \times r$.

The first step towards obtaining a solution is to eliminate $\underline{y}$ giving

$$\frac{d\underline{x}}{dt} = (A_1 - B_1B_2^{-1}A_2)\underline{x} + (C_1 - B_1B_2^{-1}C_2)\underline{z}$$

or more simply

$$\frac{d\underline{x}}{dt} = A\underline{x} + C\underline{z} \qquad (2.2)$$

This equations corresponds in control theory terminology to a *forced dynamic system*, and to obtain the solution we first consider the equivalent *free dynamic system*

$$\frac{d\underline{x}}{dt} = A\underline{x}. \tag{2.3}$$

Now let the characteristic roots of A (given by solution of $|A - sI| = 0$) be $\lambda_1, \lambda_2, \ldots, \lambda_k$ and let the corresponding characteristic vectors be $P_1, P_2, \ldots, P_k$ which we will assume to be linearly independent. Then by definition

$$AP_i = \lambda_i P_i \quad \text{for } i = 1 \to k$$

or writing $P = [P_1 \quad P_2 \quad \ldots \quad P_k]$ that is

$$AP = DP, \tag{2.4}$$

where D is the diagonal matrix

$$D = \begin{vmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_k \end{vmatrix}.$$

Now since the vectors P_i $(i = 1 \to k)$ are linearly independent then P is non-singular and hence eqn. *(2.4)* may be rewritten

$$P^{-1}AP = D. \tag{2.4a}$$

Now if a solution to eqn. *(2.3)* is required let

$$\underline{x} = P\underline{u}$$

and substitute in the equation to give

$$P\frac{d\underline{u}}{dt} = AP\underline{u}$$

or

$$\frac{d\underline{u}}{dt} = P^{-1}AP\underline{u} = D\underline{u}.$$

Thus the system of k simultaneous first order differential equations represented by eqn. *(2.3)* has been transformed into k first order differential equations of the form

$$\frac{du_i}{dt} = \lambda_i u_i \quad (i = 1 \rightarrow k)$$

which can be solved independently of each other giving

$$\underline{u} = Q(t)\underline{u}(0)$$

where $\underline{u}(0)$ is the initial value of $\underline{u}$ at $t = 0$ and

$$Q(t) = \begin{vmatrix} exp\{\lambda_1 t\} & & & 0 \\ & exp\{\lambda_2 t\} & & \\ & & \ddots & \\ 0 & & & exp\{\lambda_k t\} \end{vmatrix}.$$

Transforming the coordinates back to $\underline{x}$ the final solution of the free system is given by

$$P^{-1}\underline{x} = Q(t)P^{-1}\underline{x}(0)$$

or $$\underline{x} = PQ(t)P^{-1}\underline{x}(0). \qquad (2.5)$$

Now if the matrix C is non-zero in eqn. *(2.2)* the solution to the forced system is obtained as follows. Again substitute $\underline{x} = P\underline{u}$ to give

$$\frac{d\underline{u}}{dt} = P^{-1}AP\underline{u} + P^{-1}C\underline{z},$$

or $$\frac{d\underline{u}}{dt} - D\underline{u} = P^{-1}C\underline{z}.$$

This equation can also be solved separately for $u_1, u_2, \ldots, u_k$ but retaining the matrix notation and multiplying by the integrating factor which is Q^{-1}

$$Q^{-1}\frac{d\underline{u}}{dt} - Q^{-1}D\underline{u} = Q^{-1}P^{-1}C\underline{z}$$

or $$\frac{d}{dt}(Q^{-1}\underline{u}) = Q^{-1}P^{-1}C\underline{z}.$$

Integrating this last equation between 0 and t we obtain

$$Q^{-1}\underline{u} - \underline{u}(0) = \int_0^t Q^{-1}(\tau)P^{-1}C\underline{z}(\tau)d\tau$$

that is

$$\underline{u} = Q(t)\underline{u}(0) + Q(t)\int_0^t Q^{-1}(\tau)P^{-1}C\underline{z}(\tau)d\tau$$

or

$$\underline{x} = PQ(t)P^{-1}\,\underline{x}(0) + PQ(t)\int_0^t Q^{-1}(\tau)P^{-1}Cz(\tau)d\tau. \qquad (2.6)$$

This is the solution to eqn. *(2.2)*.

In the case where A has less than k independent characteristic vectors which may occur with multiple roots, the derivation is similar although the diagonal matrix D must be replaced by a Jordan Canonical form (see Ogata[48] for further details).

Thus it is that the solution and characteristics of the linear model are easily obtainable. The free system may exhibit exponential growth or decay or, if some of the characteristic roots of A

are complex, oscillatory motion which one might expect to be damped in any feasible economic system. Additionally the endogenous variables may force other dominant characteristics on the complete model.

Naturally this well developed theory for linear systems is an attractive reason for adopting such a model, but on the other hand it is obviously relevant to question the *adequacy* of the linear model as a representation of a national economic system (when the latter will clearly possess non-linearities of numerous and varied kinds). It is possible however with linear models like those exampled below, not only to obtain analytic solutions, but also as a consequence to derive quite meaningful results in terms of economic equilibrium dynamics and to explore problems such as capital formation, utilisation of capacity and balance of payments. The linear model therefore can fulfil a conceptual role both as a first-stage test bed to study means of implementing policy through external (government) control, and as a foundation upon which more sophisticated non-linear models can be constructed.

2.1 An Example of a Simple Linear Growth Model (Model I)

According to standard practice the following variables are defined:

$K(t)$ Stock of capital goods and assets (the term *plant* is also a useful description),

$I(t)$ Current investment in new capital,

$Y(t)$ Gross national product (GNP),

$C(t)$ All consumption on goods and services,

$M(t)$ Imports of goods and services,

$X(t)$ Exports of goods and services.

For a fuller discussion of the definition of these terms and of their measure the reader is referred to a general economic text such as Lipsey[41] or Samuelson[57] and/or Central Statistical Office[11].

To fix ideas it is convenient to think in terms of a time interval of one year, thus for example Y is the annual GNP. Also the concern here, at least for the time being, is with real values at constant prices.

In the present model a single national accounting identity (comparable with eqn. *(1.1)*) equates the national product with gross national income which has the three components of consumption, investment and net exports,

$$Y = C + I + (X - M). \tag{2.7}$$

In the absence of capital depreciation (an assumption which will be questioned later) the capital stock is the total accumulated value of investment flows which may be written

$$K = \int_{-\infty}^{t} I d\tau \quad \text{or} \quad \frac{dK}{dt} = I. \tag{2.8}$$

Next let us take imports as a constant proportion, p of the national product on the assumption that at least the demand for raw materials if not also finished goods from the foreign sector will rise with the national product

$$M = pY. \tag{2.9}$$

Now in a growing economy capital stock must grow to cater for the increase in production and so let

$$\frac{dK}{dt} = g_K K. \tag{2.10}$$

Further, in such an economy workers or households will not be satisfied if they do not share in the growing wealth; and with increased wealth the demand for consumer goods will grow. Again let us assume proportionate growth so that

$$\frac{dC}{dt} = g_C C \tag{2.11}$$

where like g_K, g_C is also a constant.

Finally, for the one remaining degree of freedom in this six variable model, consider the growth in exports. In addition to an exponential growth term (the more the home economy produces, the more it might expect to export) a built-in control mechanism is postulated so that the analysis may be potentially more fruitful. The idea here is that to produce one unit of output necessitates the utilisation of σ units of capital stock or plant. The capital-output ratio, σ is approximately constant for economies even as they advance with technical progress. Therefore K/σ is the total productive capacity of the economy, only equal to Y under conditions of full employment of resources. Generally one might expect an excess of capacity of $K/\sigma - Y$ in the economy, which the greater it becomes may cause the producer to look further afield for new markets; exports can be expected to grow. The last equation of the model makes the rate of increase of exports proportional to both the level of exports and excess capacity,

$$\frac{dX}{dt} = g_X X + q(K/\sigma - Y) \qquad (2.12)$$

subject to $K/\sigma - Y \geq 0$. The model is thus complete and in terms of the generalised form represented by eqn. *(2.1)* $\underline{x} = [K, C, X]'$, $\underline{y} = [I, Y, M]'$ and there are no exogenous variables.

However in this simple case eqns. *(2.10)* and *(2.11)* can be solved independently to give

$$K = K_0 \ exp\{g_K t\}$$

and $\quad C = C_0 \ exp\{g_C t\}$,

and then substitution for C, I and M in eqn. *(2.7)* gives

$$\frac{dX}{dt} = g_X X + A \ exp\{g_K t\} + B \ exp\{g_C t\} - qX/(1 + p),$$

where $A = qK_0 [1/\sigma - g_K/(1 + p)]$

and $\quad B = - qC_0/(1 + p)$.

Also writing $g_{XX} = g_X - \frac{q}{1 + p}$, that is

$$\frac{dX}{dt} - g_{XX} X = A \ exp\{g_K t\} + B \ exp\{g_C t\}.$$

Hence the solution for X is

$$X = X_0 \ exp\{g_{XX} t\} + \frac{A}{g_K - g_{XX}} exp\{g_K t\} + \frac{B}{g_C - g_{XX}} exp\{g_C t\}$$

if $\quad g_K \neq g_{XX}$ and $g_C \neq g_{XX}$. $\qquad (2.13)$

Alternative solutions are

$$X = (X_0 + At)\ exp\{gt\} + \frac{B}{g_C - g}\ exp\{g_C t\} \qquad (2.14)$$

if $g_{XX} = g_K$ (= g, say) and $g_C \neq g$; or

$$X = (X_0 + Bt)\ exp\{gt\} + \frac{A}{g_K - g}\ exp\{g_K t\} \qquad (2.15)$$

if $g_{XX} = g_C$ (= g, say) and $g_K \neq g$; or

$$X = (X_0 + At + Bt)\ exp\{gt\} \qquad (2.16)$$

if $g_{XX} = g_C = g_K$ (= g say).

2.2 Constraints on Growth

In order to interpret the above solution, consider the following objectives which a government may wish to pursue:

(a) to obtain the maximum possible growth rate in consumption;

(b) to obtain a long term balance on the foreign trade account, that is $X - M = 0$ asymptotically; and

(c) to retain a ceiling for excess capacity of $100\ell\%$ of total capacity, that is $0 \leq 1 - \sigma Y/K \leq \ell$.

The characteristic roots of the model are g_C, g_K and g_{XX}. We must clearly assume that both g_C and g_K are non-negative, but for g_{XX} this will not necessarily be the case. If the built-in control in eqn. *(2.12)* is sufficiently strong so that $q > (1 + p)g_X$ then the third root will be negative.

In the case of distinct characteristic roots, the expressions for the foreign trade and excess capacity are obtained as follows.

From eqn. (2.7)

$$Y + M = C + I + X,$$

that is

$$\begin{aligned}(1 + p)Y &= C_0 \exp\{g_C t\} + g_K K_0 \exp\{g_K t\} + X \\ &= X_0 \exp\{g_{XX} t\} + \left(\frac{A}{g_K - g_{XX}} + g_K K_0\right) \exp\{g_K t\} \\ &\quad + \left(\frac{B}{g_C - g_{XX}} + C_0\right) \exp\{g_C t\}.\end{aligned}$$

Consequently

$$\begin{aligned}&X - M = X - pY = \\ &\left[X_0 \exp\{g_{XX} t\} + \left(\frac{A}{g_K - g_{XX}} - p g_K K_0\right) \exp\{g_K t\}\right. \\ &\quad \left. + \left(\frac{B}{g_C - g_{XX}} - p C_0\right) \exp\{g_C t\}\right]/(1 + p)\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}&1 - \sigma Y/K = 1 - \frac{\sigma}{K_0(1 + p)} \times \\ &\left[X_0 \exp\{(g_{XX} - g_K)t\} + \left(\frac{B}{g_C - g_{XX}} + C_0\right) \exp\{(g_C - g_K)t\}\right. \\ &\quad \left. + \left(\frac{A}{g_K - g_{XX}} + g_K K_0\right)\right].\end{aligned} \tag{2.18}$$

In attempting to match these results against the objectives the cases $g_C > g_K$, $g_C = g_K$ and $g_C < g_K$ will be considered separately.

2.2.1 Case $g_C > g_K$

If $g_{XX} < g_C$ then to satisfy the balance of payments criterion we must have, considering the dominant exponential term in eqn. (2.17),

$$\frac{B}{g_C - g_{XX}} - pC_0 = 0, \tag{2.19}$$

and also to maintain excess capacity within bounds it follows similarly from eqn. *(2.18)* that

$$\frac{B}{g_C - g_{XX}} + C_0 = 0. \tag{2.20}$$

Eqns. *(2.19)* and *(2.20)* are contradictory so next consider the case in which $g_{XX} > g_C$ $(> g_K)$. Under these circumstances the coefficient of the dominant term in the balance of trade expression is $X_0/(1 + p)$ and in general this is non-zero. Again the objectives cannot be met.

Lastly consider the case of $g_{XX} = g_C$ where eqns. *(2.19)* and *(2.20)* take on the modified forms (referring back to eqn. *(2.15)* for this special equal roots case) of

$$X_0 + Bt - pC_0 = 0,$$

and $\quad X_0 + Bt + C_0 = 0$

respectively, which again are contradictory. We thus arrive at the important conclusion that the growth rate of consumption cannot exceed the growth rate of investment without leading to a balance of payments crisis and/or a shortage of capacity.

2.2.2 Case $g_C = g_K$ ($= g$, say)

In this case eqns. *(2.17)* and *(2.18)* are somewhat simplified. If $g_{XX} < 0$ then the balance of payments criterion requires, from eqn. *(2.17)*, only that

$$\left(\frac{A}{g - g_{XX}} - pgK_0\right) + \left(\frac{B}{g - g_{XX}} - pC_0\right) = 0.$$

or

$$(A + B) = (g - g_{XX})p(C_0 + gK_0). \qquad (2.21)$$

Turning to excess capacity the first term in eqn. *(2.18)* can be ignored as it is asymptotically zero and the remainder of the expression gives by applying criterion *(c)*.

$$0 \leq 1 - \frac{\sigma}{K_0(1 + p)} \left[\left(\frac{B}{g - g_{XX}} + C_0\right) + \left(\frac{A}{g - g_{XX}} + gK_0\right)\right] \leq \ell$$

or substituting for $(A + B)$ from above

$$0 \leq 1 - \frac{\sigma}{K_0} (C_0 + gK_0) \leq \ell$$

which gives

$$(1 - \ell)/\sigma - C_0/K_0 \leq g \leq 1/\sigma - C_0/K_0. \qquad (2.22)$$

These are bounds for an acceptable growth rate in consumption and investments. Whilst the width of these bounds is governed by the acceptable level of excess capacity, the maximum growth rate is (not surprisingly) synonymous with full utilisation of capacity in which case exports must also grow at the same rate, because substituting for $(A + B)$ in eqn. *(2.21)*

$$q[K_0/\sigma - (C_0 + gK_0)/(1 + p)] = (g - g_{XX})p(C_0 + gK_0)$$

where $g = g_{max} = 1/\sigma - C_0/K_0$ or equivalently $(C_0 + gK_0) = K_0/\sigma$ from eqn. *(2.22)*. Hence

$$qK_0/\sigma(1 - \frac{1}{1 + p}) = (g - g_{XX})pK_0/\sigma$$

that is

$$\frac{q}{1 + p} = g - g_{XX}$$

or $$g_X = g_{XX} + \frac{q}{1 + p} = g.$$

If $g_{XX} > 0$ then the balance of payments criterion will be violated by exponential growth of the first term in eqn. *(2.17)* unless $g_{XX} = g$. In this latter case (referring this time to eqn. *(2.16)*) the equivalent of eqn. *(2.21)* is

$$X_0 + (At - pgK_0) + (Bt - pC_0) = 0 \quad \text{as } t \to \infty$$

which implies that

$$\left.\begin{aligned} &A + B = 0 \\ \text{and} \quad &X_0 = p(C_0 + gK_0) \end{aligned}\right\}$$

simultaneously. But in general

$$\begin{aligned} A + B &= q[K_0/\sigma - (C_0 + gK_0)/(1 + p)] \\ &\doteq q[K_0/\sigma - X_0/(p + p^2)] \neq 0 \quad \text{for } q \neq 0 \end{aligned}$$

and hence the objectives cannot be met.

2.2.3 Case $g_C < g_K$

It remains to be seen in this third case whether there is any gain in investing at a higher rate than the rate of consumption. To maintain excess capacity within bounds the first term in eqn. *(2.18)* must be damped which implies that $g_{XX} \leq g_K$. Now if $g_{XX} < 0$ the balance of payments criterion requires that

$$\frac{A}{g_K - g_{XX}} - pg_K K_0 = \frac{B}{g_C - g_{XX}} - pC_0 = 0 \tag{2.23}$$

and the excess capacity constraint is

$$0 \leq 1 - \frac{\sigma}{K_0(1 + p)} \left(\frac{A}{g_K - g_{XX}} + g_K K_0\right) \leq \ell$$

i.e. $0 \leq 1 - g_K\sigma \leq \ell$

or $\quad (1 - \ell)/\sigma \leq g_K \leq 1/\sigma. \qquad (2.24)$

Eqns. (2.23) then determine appropriate growth rates for g_X and g_C. In particular eliminating g_{XX}

$$g_K - \frac{A}{pg_K K_0} = g_C - \frac{B}{pC_0},$$

and substituting for A and B

$$g_C = g_K - \frac{q}{\sigma p g_K}.$$

The bounds for g_C are thus

$$(1 - \ell)\sigma - q/[p(1 - \ell)] \leq g_C \leq 1/\sigma - q/p. \tag{2.25}$$

Once again the maximum growth rate is synonymous with full capacity utilisation and comparing with the similar case in Sec. 2.2.2 where the maximum growth rate for consumption was $1/\sigma - C_0/K_0$, the present policy is superior only if

$$C_0/K_0 > q/p.$$

C_0/K_0 is a measure of some initial state of the economy, and in particular would be greater if the economy were initially in an unhealthy state (as characterised by a high level of consumption relative to a low level of stock). Under these circumstances the present policy might be preferable.

If $g_{XX} > 0$ the balance of payments criterion can only be satisfied if $g_{XX} = g_C$ or $g_{XX} = g_K$. In the former case we must have

$$X_0 + Bt - pC_0 = 0 \quad \text{as} \quad t \to \infty$$

which is not possible as $B \neq 0$, and in the latter case

$$X_0 + At - pg_K K_0 = 0 \quad \text{as} \quad t \to \infty.$$

The last equation requires that $A = 0$ and $X_0 = pg_K K_0$ simultaneously and these two conditions in general imply two different values for g_K. Again the objectives cannot be met if $g_{XX} > 0$.

To summarise the conclusions of the present section it has been shown that:

(i) the rate of growth of consumption cannot exceed the rate of growth of investment;

(ii) in meeting the objectives which are specified at the outset of this section it is important that any excess capacity be absorbed by a rapid expansion of exports so that $q > (1 + p)g_X$;

(iii) given (ii) it may be advantageous in certain circumstances to expand investment at a greater rate than consumption.

Thus it is demonstrated that model I, despite its utter simplicity, does produce evidently meaningful and intuitively reasonable results. There is clearly scope for further study on this basic model by, for example, identifying a feedback effect in the equations which determine consumption and investment again considering their sensitivity to variations in excess capacity. We choose however to take an alternative course, disaggregating the model into separate accounts for households, government, production and capital stock thus demonstrating how insight may be gained to the internal flows of money such as savings and taxes - a fundamental instrument of policy implementation.

2.3 An Expanded Model (Model IA)

The following expanded model was first proposed by Parks and Pyatt[50, 51] and the increased detail can best be defined in an accounting matrix as shown in fig. 2. The new variables which have been introduced are:

$G(t)$ Government expenditure on goods and services,

$T_h(t)$ Government income from taxes levied on private households,

$T_c(t)$ Government income from taxes levied on companies,

$W(t)$ Gross national wage bill paid to households,

$D(t)$ Income of private households from investment dividends,

$R(t)$ Total before tax profits on company activities,

$I_f(t)$ Investment in foreign accounts,

$S_g(t)$, $S_h(t)$, $S_c(t)$ Savings of government, households and companies respectively.

In addition $C(t)$ is amended in the present model to exclude government spending and $I(t)$ to exclude foreign investment.

The entries recorded in the accounting matrix all refer to money flows. Money received by an account appears in the row for

		EXPENDITURES					
		prod.	hous.	govn.	comp.	cap.	r.o.w.
RECEIPTS	production	—	C	G	•	I	X
	households	W	—	•	D	•	•
	government	•	T_h	—	T_c	•	•
	companies	R	•	•	—	•	•
	capital	•	S_h	S_g	S_c	—	•
	rest of world	M	•	•	•	I_f	—

FIGURE 2 Social accounting matrix for model IA

that account; money spent appears in the corresponding column. By accounting convention the diagonal elements of the matrix are omitted as transactions within accounts are of no interest.

The matrix shown in fig. 2 distinguishes six accounts. The first is the production account which is the consolidated trading account of all production activities. The next three accounts are the current accounts of the domestic institutions, namely households, companies and government. The fifth account is a consolidated capital account for all domestic institutions, and the last account is the combined current and capital account of the rest of the world.

Tracing through the accounts, we see that production activities receive money from households in payment for consumer goods C, from government on account of its current expenditure G, and also from the last two accounts for capital and exported goods respectively. On the expenditure side, after payment of wages W and for imports M, the balancing item of profits R is paid to companies. The identity of receipts and expenditure on the production account leads to the expanded version of eqn. *(2.7)*.

Households receive income from company dividends D in addition to wages W. After spending C on consumer goods and paying taxes T_h to the government, their remaining income is saved by payment of S_h to the combined capital account.

All government income is made up from taxes paid by households T_h and companies T_c. Expenditure on goods and services G goes to the production activities and the balancing item here is government savings S_g. S_g can be negative if there is a budget deficit.

The only source of company income is profits R from domestic production. Some of this income is paid out to households as distributed profits D and to government in the form of taxes T_c. The balancing residual is S_c, which is company savings.

At this point, we see that the combined domestic capital account has three sources of income, which are the savings of the three domestic institutions. This money is invested at home by buying capital goods I or goes abroad as foreign investment I_f.

The balance of payments surplus in current transactions with the rest of the world is represented by the excess of X over M. This is offset in the final account by foreign investment I_f.

Thus the accounting system is specified and the fact that for each account the total receipts and total expenditure must be equal leads to the first five independent equations of the model (the sixth equation is redundant because the system is closed). If however we adopt all the equations of model I there are still five degrees of freedom to be removed in order that the new model shall be determinate. Let us assume that the manner in which taxes are determined may be stipulated as follows:

$$T_h = t_h(W + D), \tag{2.26}$$

and $$T_c = t_c R - t_a I. \tag{2.27}$$

That is households are taxed in proportion to their total income and companies are taxed in proportion to their profits with an allowance given for investments. In this still simple model taxes are applied linearly and the taxation indices t_h, t_c and t_a are all constant.

Now consider the following hypotheses as some alternative ways of completing the model:

(i) a policy for the determination of the division of national income between wages and profits;

(ii) a policy for the division of company after tax profits as between reinvestment or distribution to households in the form of dividends;

(iii) policies for the determination of saving or spending by households and government;

(iv) a policy of zero foreign trade imbalance;

(v) a policy of zero excess capacity.

To adopt any three of the foregoing would make the model fully determinate enabling a solution to be found. As an example Hulme[34] in some work on such a model adopted the following determining set of equations:

$$W = a_1(W + R) + a_2[K/\sigma - (W + R)], \tag{2.28}$$

wages are here set as a proportion of the domestic product plus an amount which increases with excess capacity;

$$Sc = bI, \tag{2.29}$$

it is the policy of companies to save in proportion to their total investment requirement; and

$$X = M \quad \text{or} \quad I_f = 0, \tag{2.30}$$

there is no foreign trade imbalance. Finally instead of eqns.

(2.10) and *(2.11)* Hulme makes the following assumptions regarding the growth of household and public spending,

$$\frac{dC}{dt} = c_1(C + S_h) + c_2C, \tag{2.31}$$

and

$$\frac{dG}{dt} = gG + d[K/\sigma - (W + R)]. \tag{2.32}$$

The equation for determining household consumption is similar to eqn. *(2.11)* but government expenditure is now made sensitive to excess capacity. Hulme was investigating the latter as an instrument of government control; and in fact in terms of the assumptions built into the model he was able to demonstrate that this means provided only a relatively unresponsive control instrument (for further details the reader is referred to the original work).

2.4 Sensitivity of the Model to Changes in Taxation

In a similar manner to Hulme, we now demonstrate how the effectiveness of the taxes themselves can be examined in the expanded model. The system equations remain unchanged except that d is set to zero in eqn. *(2.32)* and G henceforth grows at an exponential rate g. Fig. 3 shows the model equations in matrix form.

The following elementary row operations are now applied to reduce the matrix equation to a four dimensional system in the vector $[K, C, X, G]'$:

OURCE	K	C	X	G	S_h	W	R	M	D	T_h	T_c	S_c	S_g	I		ROW NUMBE
qn. (2.8)	$\frac{d}{dt}$	·	·	·	·	·	·	·	·	·	·	·	·	-1		(1)
qn. (2.31)	·	$\frac{d}{dt}-(c_1+c_2)$	·	·	$-c_1$	·	·	·	·	·	·	·	·	·		(2)
qn. (2.12)	$-q/\sigma$	·	$\frac{d}{dt}-g_x$	·	·	q	q	·	·	·	·	·	·	·		(3)
qn. (2.32)	·	·	·	$\frac{d}{dt}-g$	·	·	·	·	·	·	·	·	·	·		(4)
	·	1	1	1	·	-1	-1	-1	·	·	·	·	·	1		(5)
ational	·	–	·	·	–	1	·	·	1	-1	·	·	·	·		(6)
ccounting	·	·	·	-1	·	·	·	·	·	1	1	·	-1	·		(7)
identities	·	·	·	·	·	·	1	·	-1	·	-1	-1	·	·	$=0$	(8)
	·	·	·	·	1	·	·	·	·	·	·	1	1	-1		(9)
eqn. (2.9)	·	·	1	·	·	$-p$	$-p$	·	·	·	·	·	·	·		(10)
eqn. (2.26)	·	·	·	·	·	$-t_h$	·	·	$-t_h$	1	·	·	·	·		(11)
eqn. (2.27)	·	·	·	·	·	·	$-t_c$	·	·	·	1	·	·	t_a		(12)
eqn. (2.28)	$-a_2/\sigma$	·	·	·	·	$1-a_1+a_2$	$-a_1+a_2$	·	·	·	·	·	·	·		(13)
eqn. (2.29)	·	·	·	·	·	·	·	·	·	·	·	1	·	$-b$		(14)

FIGURE 3 Matrix equation for model IA

$R8 + R14$ and $R9 - R14$ to eliminate S_c;

$R6 + R11$ and $R7 - R11$ to eliminate T_h;

$R7 - R12$ and $R8 + R12$ to eliminate T_c;

$R7 + R9$ then eliminates S_g;

$R7 + R6$, $R7 + R8$ and $R6 + (1 - t_h) \times R8$ to eliminate D;

$R10 \div p$, $R3 + q \times R10$, $R6 + (1 - t_h) \times R10$, $R7 + R10$ and $R13 + (1 - a_1 + a_2) \times R10$ to eliminate W;

$R6 + t_c(1 - t_h) \times R13$ to eliminate R;

$R1 - R7$ and $R6 + (1 - t_h)(t_a - b)R7$ to eliminate I;

and finally $R2 - C_1 \times R6$ to eliminate S_h.

This reduction then leads to a characteristic equation for the state vector,

$$\begin{vmatrix} s & 1 & -1/p & 1 \\ -a_2 t_c/\sigma & \dfrac{1}{c_1}(s-c_2)/(1-t_h) + (t_a-b) & -\dfrac{1}{p}(t_a-b) - \dfrac{1}{p}[1-t_c(1-a_1+a_2)] & (t_a-b) \\ -q/\sigma & 0 & s+q/p-g_X & 0 \\ 0 & 0 & 0 & s-g \end{vmatrix} = 0 \tag{2.33}$$

The influence of the various tax indices may now be examined by root locus analysis (see Ogata[49]), subject to the adoption of appropriate values for the parameters. Values similar to those originally suggested by Parks and Pyatt and as amended by Hulme are used and are as follows:

$a_1 = 0{\cdot}599$ $a_2 = 0{\cdot}197$

$c_1 = 0{\cdot}3$ $c_2 = -0{\cdot}26$

$p = 0{\cdot}232$ $q = 0{\cdot}00696$

$\sigma = 3$ $b = 0{\cdot}6$

$g_X = 0{\cdot}03$ $t_h = 0{\cdot}1575$

$t_c = 0{\cdot}4$ $t_a = 0{\cdot}1$

Substitution of these values into eqn. *(2.33)* yields one positive real characteristic root of $0{\cdot}067$ and two roots with negative real part in addition to the root g, the positive growth rate of government expenditure. Hulme's parameter estimates were based on Central Statistical Office data[10] and accordingly the results are intended to bear some reflection on the U.K. economy. It should also be pointed out that Hulme used current price data and therefore, although an asymptotic growth rate of 6·7% per annum may appear large, it is intended to include the effect of price inflation (in fact the average growth rate of GNP for the period 1960–70 was 6·34% which in real terms was 3·24%). The actual value of g is immaterial to the present analysis but in view of the constraints on growth derived in the previous section it is clearly desirable that $g \leq 6{\cdot}7\%$ unless the asymptotic growth rate of $6{\cdot}7\%$ can be raised.

We now consider how the asymptotic growth rate of the economy is affected by manipulation of the taxation indices t_h and t_c defined by eqns. *(2.26)* and *(2.27)*. The most immediate way of assessing this effect is to draw root locus plots for each of the taxation indices. To obtain the root locus in the s-plane for the parameter k the characteristic equation (in this case eqn. *(2.33)*) is written in the form

$$p(s) + kq(s) = 0$$

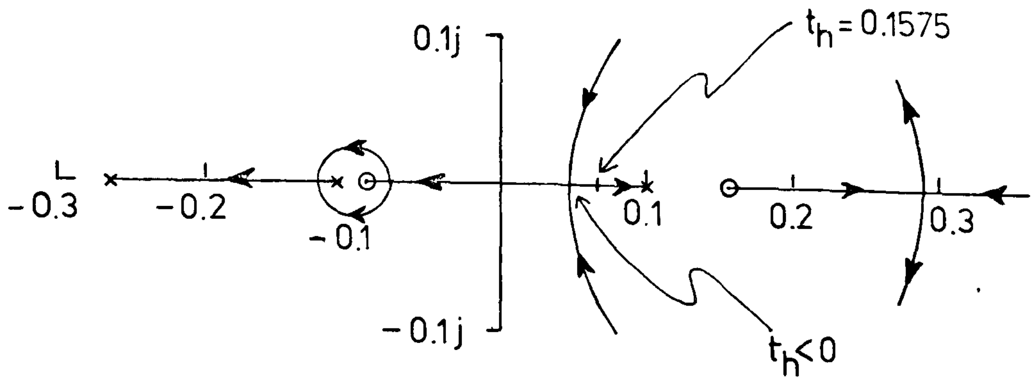
or $$1 + k\frac{q(s)}{p(s)} = 0$$

where $p(s)$ and $q(s)$ are polynomial in s. Now when $k = 0$ the roots of the characteristic equation are given by $p(s) = 0$ and are known as the poles. The branches of the root locus, which originate at the poles, are then given by increasing k. In the limit as $k \to \infty$ the roots are given by $q(s) = 0$ and are known as the zeros.

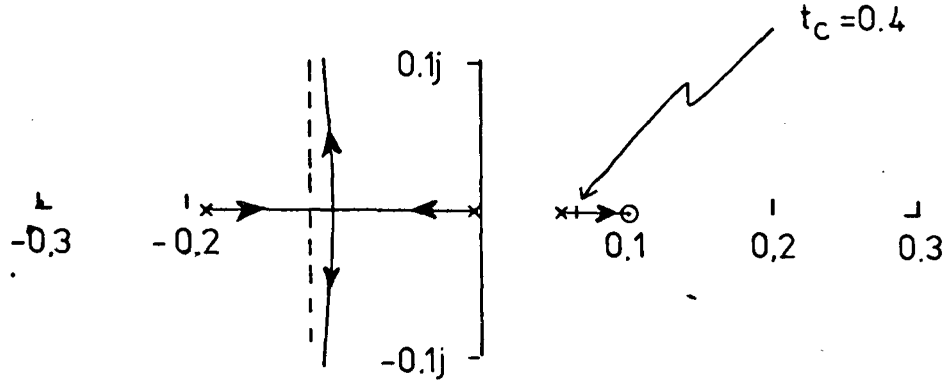
Fig. 4 (a) is the root locus plot for $k = 1 - t_h$ where the poles and zeros are represented by crosses and circles respectively. Fig. 4 (b) is similarly the root locus plot for $k = t_c$ and in each case the arrows show the effect of an increase in the respective taxation index.

Apart from the root g eqn. *(2.33)* has in general three other roots so that in fig. 4 (a) there are for instance two poles which are real and negative and a third at $s = 0{\cdot}10$ (the poles correspond to $t_h = 1$). As $1 - t_h$ is increased to infinity (t_h decreased) the positive root remains real but decreases taking the value $s = 0{\cdot}067$ at the estimated value of t_h ($= 0{\cdot}1575$), and not reaching the branch point until t_h is negative. Exponential growth is thus assured for $0 < t_h < 1$. Similarly in fig. 4 (b) the one positive root is real and moves from a pole at $s = 0{\cdot}057$ to a zero at $s = 0{\cdot}10$ as increases from 0 to ∞. Hence it would appear that an increase in the rate of economic growth can be achieved by independent increases in either of income tax or company tax.

In contrast to traditional engineering applications of the

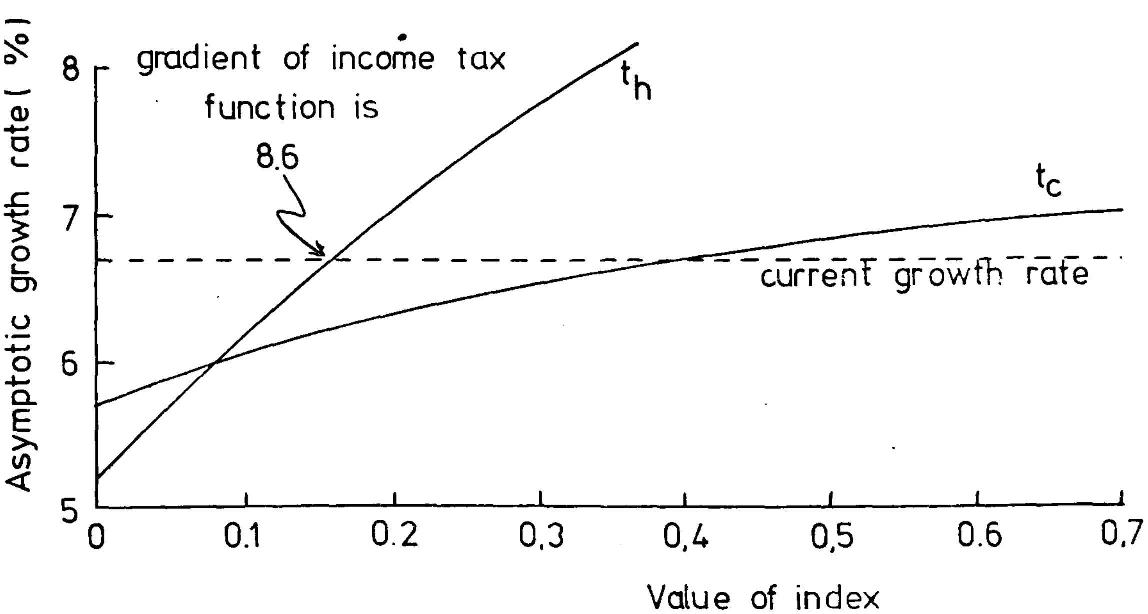


(a) Root locus for $(1-t_h)$



(b) Root locus for t_c

JRE 4 Root locus plots for model IA



;URE 5 Asymptotic growth rates for model IA at different income root locus method, its application to economic systems focuses interest on the right hand half of the complex plane and in particular on the positive real axis. In order to examine the sensitivity of the system to changes in taxation more closely, that portion of the real axis which lies between $s = 0{\cdot}05$ and $0{\cdot}08$ is magnified in fig. 5, in which plots are given for the asymptotic growth rate of the economy versus the taxation indices t_h and t_c. The *current* growth rate of $6{\cdot}7\%$ represented by the broken line is that given by the parameter values above. We can obtain a measure of the system sensitivity by taking the gradient of the respective curves at the *current* growth rate. For the household income tax curve this is about $1{\cdot}35$ relative to the value of the index; that is to say that as an approximate linear rule to double income tax would increase rate of growth by $1{\cdot}35\%$ per annum. Correspondingly it is clearly from the second curve that the system as represented by its asymptotic growth rate is very insensitive to changes in company tax.

Whilst the above results are presented largely as illustrative of linear model techniques, it is never-the-less instructive to understand how they arise in the mechanism of the model. We can observe from simulation studies on Model IA that under increased income tax the consumption of households drops, although the standard of living is maintained to some extent by a substantial reduction in the propensity of households to save. The government as the recipient of an increase in revenue becomes the agent of net extra investment, thus leading to an increase in the asymptotic rate of economic growth. Similarly companies under pressure of increased taxation, whilst maintaining the savings policy of eqn. *(2.29)* will be forced to reduce dividends. Households suffer this cut in income much as if tax is increased although we have already seen that the effect is not very severe.

Thus in this brief excursion with a disaggregated version of the basic linear model it is possible to demonstrate the potential importance of control theory and systems techniques in one aspect of macroeconomics. However at this point we revert to the basic model to focus attention on the phenomenon of cyclical growth which is to be the main theme of subsequent models.

CHAPTER 3

ELEMENTARY NON - LINEAR MODELS

As previously suggested, the introduction of effective controls in a national economic system demands an understanding of both the economic mechanisms which prevail in that system and the way in which those mechanisms interact. However, whilst much is written in the neo-classical theory of economic growth on equilibrium solutions and conditions for their existence (witness the volume of a notable survey by Hahn and Matthews[25]), less attention has been focused on the study of such systems in disequilibrium. The reason for this state of affairs is not only that disequilibrium adjustment mechanisms are difficult to establish, but more significantly that the latter are likely to lead to mathematical models of rapidly increasing complexity. Never-the-less, if such a system has been disturbed from equilibrium (as will invariably be the case), or if a new equilibrium path is sought, then information on the dynamics of disequilibrium becomes necessary.

The inherent stability properties of the growth paths of national economic systems raises a fundamental question which as yet remains unanswered. An apparently most plausible argument suggests that such systems are likely to be stable having survived broadly intact after many years of existence. This opinion holds that oscillations and fluctuations have been due to random disturbances, delays in applying corrective action and possibly, in some cases, in the implementation of totally inappropriate action. The models which follow in this and subsequent chapters are developed in an attempt to tackle some of these questions.

3.1 A Simple Model of Cyclical Growth (Model II)

Whilst the models of the previous chapter have embodied growth somewhat by a leap of faith, we now look to technological progress and population growth as specific causal factors. Labour employment and unemployment subsequently feature in all models not only as stimuli for growth but also as key indicators of the state of the economy. Thus, whilst a steady rate of unemployment can be eliminated in the long term by reducing the average working hours of those employed, fluctuations relative to the steady rate are socially undesirable and reflect a state of boom or depression. The model immediately becomes non-linear and the following new variables are introduced:

$a(t) = a_0 exp\{\alpha t\}$,	the average productivity of labour,
$L(t) = L_0 exp\{\beta t\}$,	labour supply,
$E(t) = Y/a$,	total labour employment,
$W(t) = wE$,	the national wage bill where
$w(t)$	is the per capita wage rate.

The present model, believed to be due originally to Goodwin, has most in common with model I and as such the single national accounting identity

$$Y = C + I$$

describes an economy which is closed (no foreign trade) and in which the activities of the government are not singled out. Furthermore we shall assume that all wages are consumed and that all profits are invested (this is the case of *classical savings*). Substitution in the accounting identity gives

$$Y = W + I,$$

or $$I = (1 - \frac{w}{a})Y. \tag{3.1}$$

Taking a constant capital-output ratio as in the previous models, the absence of excess capacity leads to the most elementary of production functions

$$Y = K/\sigma, \tag{3.2}$$

where the economy is assumed to be operating at full capacity of capital. Using eqn. *(2.2)* to eliminate I we obtain

$$\frac{dK}{dt} = \sigma \frac{dY}{dt} = (1 - \frac{w}{a})Y. \tag{3.3}$$

Lastly for present purposes it is assumed that the rate of growth of the wage rate is related to relative employment by the linear relation

$$\frac{1}{w}\frac{dw}{dt} = -\gamma + \rho \frac{E}{L} . \tag{3.4}$$

The solution is facilitated by introducing non-dimensional variables

$$\left.\begin{aligned} u &= \frac{w}{a} = \frac{W}{Y}, \quad \text{share of wages} \\ \text{and} \quad v &= \frac{E}{L}, \quad \text{relative employment,} \end{aligned}\right\} \tag{3.5}$$

whereupon the system as represented by eqns. *(3.3)* and *(3.4)* can be reduced to these two variables.

Logarithmic differentiation of eqn. *(3.5)* gives

$$\frac{1}{u}\frac{du}{dt} = \frac{1}{w}\frac{dw}{dt} - \alpha ,$$

and $$\frac{1}{v}\frac{dv}{dt} = \frac{d}{dt}\log\left(\frac{Y}{aL}\right) = \frac{1}{Y}\frac{dY}{dt} - (\alpha + \beta)$$

whence on substitution in eqns. *(3.3)* and *(3.4)* gives

$$\frac{1}{Y}\frac{dY}{dt} = \frac{1}{v}\frac{dv}{dt} + (\alpha + \beta) = \frac{1}{\sigma}(1 - u) ,$$

and $$\frac{1}{w}\frac{dw}{dt} = \frac{1}{u}\frac{du}{dt} + \alpha = -\gamma + \rho v ;$$

i.e. $$\left.\begin{aligned} \frac{du}{dt} &= [-(\alpha + \gamma) + \rho v]u \\ \text{and} \quad \frac{dv}{dt} &= [\frac{1}{\sigma} - (\alpha + \beta) - \frac{u}{\sigma}]v. \end{aligned}\right\} \qquad (3.6)$$

The solution to eqns. *(3.6)* is obtained by firstly eliminating the term in uv to give

$$\frac{1}{\sigma}\frac{du}{dt} + \rho\frac{dv}{dt} = -\frac{1}{\sigma}Au + \rho Bv \qquad (3.7)$$

where $A = (\alpha + \gamma)$ and $B = \frac{1}{\sigma} - (\alpha + \beta)$.

But again from eqns. *(3.6)*

$$\rho v = \frac{1}{u}\frac{du}{dt} + A \text{ and } -\frac{1}{\sigma}u = \frac{1}{v}\frac{dv}{dt} - B,$$

hence substituting in turn for u and v in eqn. *(3.7)*

$$\frac{1}{\sigma}\frac{du}{dt} + \rho\frac{dv}{dt} = A\left(\frac{1}{v}\frac{dv}{dt} - B\right) + B\left(\frac{1}{u}\frac{du}{dt} + A\right),$$

i.e. $$\frac{1}{\sigma}\frac{du}{dt} + \rho\frac{dv}{dt} - A\frac{1}{v}\frac{dv}{dt} - B\frac{1}{u}\frac{du}{dt} = 0. \qquad (3.8)$$

Subsequent integration gives the solution in the form

$$\frac{1}{\sigma} u + \rho v - A \log v - B \log u = constant. \tag{3.9}$$

Fig. 6 is a phase diagram for u and v and shows the cyclical behaviour of the system over time. The initial conditions determine the value of the constant in eqn. *(3.9)* and thus the amplitude of oscillation about the equilibrium point. The equilibrium is

$$\left.\begin{aligned} \bar{u} &= 1 - \sigma(\alpha + \beta) \\ \bar{v} &= (\alpha + \gamma)/\rho \end{aligned}\right\} \tag{3.10}$$

and is neutrally stable. For a small displacement from equilibrium given by

$$\delta u = u - \bar{u} \quad \text{and} \quad \delta v = v - \bar{v} \, ,$$

the system given by eqns. *(3.6)* reduces to

$$\left.\begin{aligned} \delta u' &= \rho \delta v(\bar{u} + \delta u) \\ \delta v' &= - \frac{\delta u}{\sigma} (\bar{v} + \delta v) \end{aligned}\right\} \tag{3.11}$$

or upon linearisation to

$$\delta u'' = \rho \bar{u} \delta v' = - \frac{\rho}{\sigma} \bar{u}\bar{v} \delta u. \tag{3.12}$$

The oscillation approximates to simple harmonic motion with frequency

$$\sqrt{\frac{\rho}{\sigma} \bar{u}\bar{v}} = \sqrt{\left[\frac{1}{\sigma} - (\alpha + \beta)\right](\alpha + \gamma)} \, ,$$

and moreover the frequency is independent of ρ, the sensitivity of wages to unemployment.

Having thus achieved a meaningful result of this elementary formulation, the next section investigates the incorporation of plant with finite (limited) life.

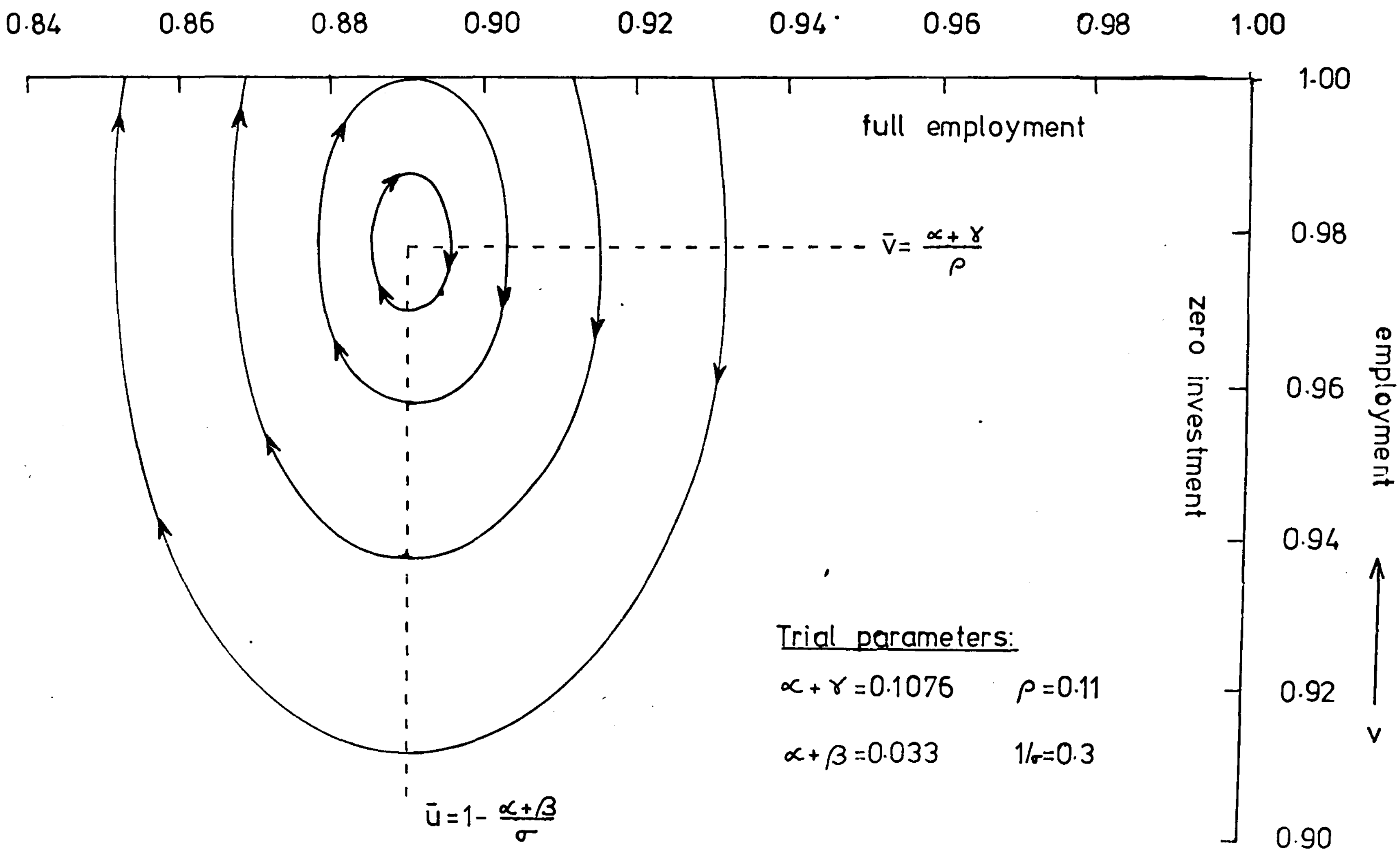


FIGURE 6 Phase diagram for model II

3.2 The Economic Life of Plant (Model IIA)

In a technologically progressive economy it is hardly reasonable to assume infinite plant life. Whilst plant may be maintained in adequate physical condition by a suitable programme of overhaul and repair, such a programme will become uneconomic as more advanced methods, materials and machines become available.

$\theta(t)$ is designated the age of the oldest plant in operation, and will be referred to as the economic life of plant. θ is determined according to the profitability of the plant in question.

When plant no longer yields a profit because of rising real wage costs the economic life of that plant is deemed to have expired. Under circumstances in which wages were forced to drop relative to prices it is feasible that previously retired plant could profitably be recommissioned. Although one might reasonably expect this occurrence to be rare, it is not specifically excluded from the present model. The profit per unit time on plant installed at time τ (subsequently referred to as plant of vintage τ) and which employs one man is equal to

$$a(\tau) - w(t)$$

at time t. Again choosing a time unit of one year, this is the annual profit per man employed on plant which is $t - \tau$ years old. The expression somewhat obscures the role of other production and raw material costs which may be considered small compared with wage costs. Alternatively and more precisely we can take the above expression as pure value added, where raw material and similar costs are passed directly to the consumer. According to the zero profit criterion then, the profit is zero when $t - \tau = \theta$ so that

$$a(t - \theta) - w(t) = 0$$

or $$a(t)\ exp\{- \alpha\theta\} - w(t) = 0$$

giving

$$\theta = \frac{1}{\alpha}\ log\left(\frac{a}{w}\right) = \frac{1}{\alpha}\ log\left(\frac{a_0}{w}\right) + t. \tag{3.13}$$

θ is thus established as a key variable closely linked to the going wage rate, w.

To specify the remainder the modified model, eqn. *(2.2)* is amended to account for finite plant life giving

$$K = \int_{t-\theta}^{t} I(\tau)d\tau \tag{3.14}$$

or in its differentiated form

$$\frac{dK}{dt} = I(t) - I(t - \theta)(1 - \frac{d\theta}{dt}). \tag{3.15}$$

Finally in model IIA we adopt a wage equation to replace eqn. *(3.4)* which is closer to the *curve* attributed to Phillips[55],

$$\frac{1}{w}\frac{dw}{dt} = -\gamma + \frac{\rho L}{U}. \tag{3.16}$$

The linear dependence upon employment is replaced by dependence upon the reciprocal of unemployment, and this modified disequilibrium wage adjustment mechanism ensures that there is always an excess of supply over demand in the labour market. Phase plane trajectories will no longer cross the full employment boundary as happened in fig. 6 for model II.

Whilst the Phillips' curve has been the centre of some controversy particularly in recent years, Lipsey[40] in a further detailed and eloquent analysis has broadly substantiated the original work. Despite its shortcomings, not the least of which is that it is based upon statistical data drawn from the period 1862 - 1957, it is adopted for the present purposes.

The model is again complete.

3.3 An Approximate Solution to Model IIA

To obtain an approximate solution to the above model it is assumed that

$$I(t - \theta) = I(t)\ exp\{-\ \alpha\theta\}.$$

That is to say, since the trend growth rate of investment is α, that in so far as investment at time $t - \theta$ is concerned fluctuations about the trend are negligible. We are assuming that investment θ years ago was at an appropriately discounted level relative to today's investment; whilst this is not the same as assuming that the economy was in steady state growth, it is an approximation of the same order. It then follows from eqn. *(3.15)* that

$$\frac{dK}{dt} = I\left[1 - exp\{-\ \alpha\theta\}(1 - \frac{d\theta}{dt})\right] .$$

But from eqn. *(3.13)*

$$exp\{-\ \alpha\theta\} = \frac{w(t)}{a(t)} \quad \text{and} \quad -\ \alpha\ \frac{d\theta}{dt} = \frac{1}{w}\frac{dw}{dt} - \alpha$$

by logarithmic differentiation, and therefore we get the important equation

$$\frac{dK}{dt} = I(1 - \frac{1}{\alpha a}\frac{dw}{dt}) \tag{3.17}$$

which defines the rate of net investment as the rate of gross investment, I, less the rate at which capital is scrapped, $I\,\frac{dw}{dt} \,/\, (\alpha a)$.

Reducing eqns. *(3.16)* and *(3.17)* to their non-dimensional form by again substituting in terms of u and v from eqn. *(3.5)* (and thus removing the exponential trend) leads to the two non-linear first order equations

$$\frac{1}{w}\;\frac{dw}{dt} = \frac{1}{u}\frac{du}{dt} + \alpha = -\gamma + \frac{\rho}{1 - v}\,,$$

and $$\sigma\,\frac{dY}{dt} = (1 - u)Y(1 - \frac{1}{\alpha a}\frac{dw}{dt})$$

or $$\frac{1}{Y}\frac{dY}{dt} = \frac{1}{v}\frac{dv}{dt} + (\alpha + \beta) = \frac{1}{\sigma}\,(1 - u)(1 - \frac{u}{\alpha w}\frac{dw}{dt})$$

$$= \frac{1}{\sigma}\,(1 - u)\left[1 - \frac{u}{\alpha}\;\;(-\gamma + \frac{\rho}{1 - v})\right].$$

i.e. $$\left.\begin{aligned} \frac{1}{u}\frac{du}{dt} &= -\;(\alpha + \beta) + \frac{\rho}{1 - v} \\ \frac{1}{v}\frac{dv}{dt} &= -\;(\alpha + \beta) + \frac{1}{\sigma}\,(1 - u)\left[1 - \frac{u}{\alpha}\,(-\gamma + \frac{\rho}{1 - v})\right]. \end{aligned}\right\} \tag{3.18}$$

Figure 7 shows the (u, v) phase plane plot for eqns. *(3.18)* with $\beta = 0$. The point of equilibrium is given by

$$\left.\begin{aligned} \bar{u} &= 1 - \sqrt{\alpha\sigma} \\ \bar{v} &= 1 - \frac{\rho}{\alpha + \gamma}. \end{aligned}\right\} \tag{3.19}$$

The equilibrium can be proved stable by again considering small displacements δu and δv. The linearised form of eqn. *(3.18)* is then

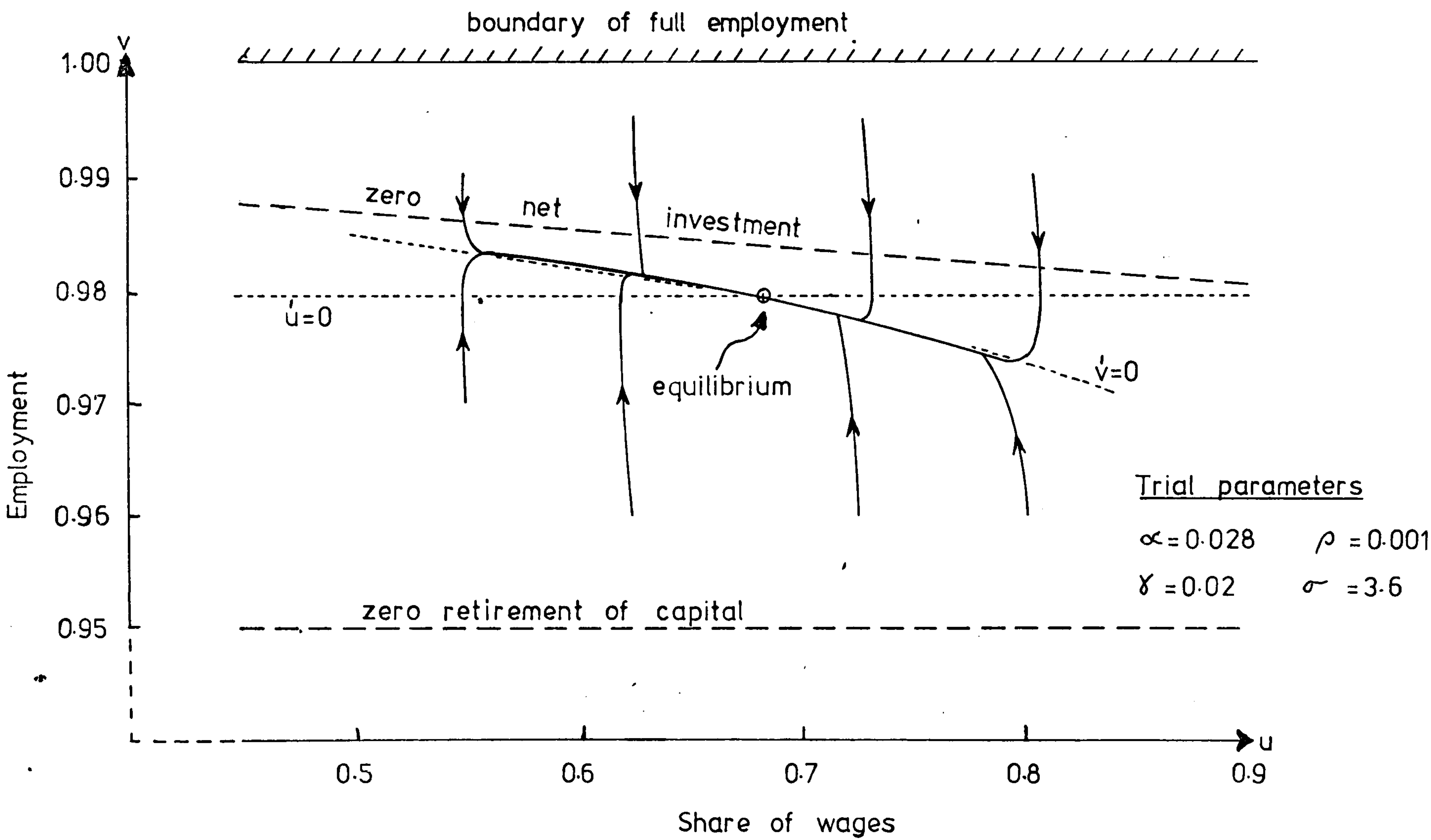


FIGURE 7 Phase diagram for model II A

$$\left.\begin{aligned} \delta\dot{u} &= \bar{u}\,\frac{\rho}{(1-\bar{v})^2}\,\delta v \\ \delta\dot{v} &= \bar{v}\,\frac{1-\bar{u}}{\sigma}\left(-\,2\delta u - \frac{1}{\alpha}\,\delta\dot{u}\right) \end{aligned}\right\} \qquad (3.20)$$

whence

$$\delta\ddot{u} = \frac{\rho}{\sigma}\,\bar{u}\,\bar{v}\,\frac{1-\bar{u}}{(1-\bar{v})^2}\left(-\,2\delta u - \frac{1}{\alpha}\,\delta\dot{u}\right). \qquad (3.21)$$

The motion is locally one of damped sinusoidal oscillations or critically damped exponential decay. The frequency of any oscillation is not now independent of ρ as it was in the previous model (although of course this is a numerically different parameter ρ).

The phase diagram was computed by digital simulation using a simple Runge-Kutta algorithm and a trial set of parameters intended to be compatible with the British experience (the reader is referred to chapter 6 where the question of appropriate parameter values is discussed). In the region of the boundary of full employment wages are rising so fast that there is net disinvestment because of the high rate of retirement of capital. This unstable situation is rapidly rectified as the trajectories fall below the boundary of zero net investment which is given by

$$1 - \frac{u}{\alpha}\left(-\,\gamma + \frac{\rho}{1-v}\right) = 0\,.$$

Here is the normal region of operation of the economy. At the boundary given by

$$-\,\gamma + \frac{\rho}{1-v} = 0$$

$\frac{dw}{dt} = 0$ by eqn. *(3.16)* and hence $\frac{d\theta}{dt} = 1$ by eqn. *(3.13)*. The vintage of the oldest plant in operation remains constant and this is the boundary of zero retirement of capital.

3.4 The Inclusion of a Price Effect (Model IIB)

The recent United Kingdom experience of rapidly rising wage levels even in a period of high unemployment has further fired the debate on the determinants of wage levels and wage inflation. It has been variously suggested that change in unemployment rather than the level of unemployment is the key factor, or that profits or prices are the major determinants. The evidence, says Kuh[39], for unemployment as the sole determinant and acceptance of the Phillips Curve *appears contradictory or at the very least not overwhelming.*

As a consequence of this situation it would seem desirable to investigate a modification to the wage equation. This is done, following Kuh, by introducing dependence of wages upon productivity. Let the index of prices in our single produce economy be P, so that Pa represents average value productivity which is included in the new structure. The relationship to prices is very close particularly as at present a is purely exponential. Eqn. *(3.4)* in the basic model (II) becomes

$$\frac{1}{w}\frac{dw}{dt} = -\gamma + \rho\frac{E}{L} + \eta\frac{1}{Pa}\frac{d}{dt}(Pa). \tag{3.22}$$

Thus, in addition to the link between wages and unemployment there is a pressure to increase wages as average value productivity rises. The new parameter η determines the extent to which this pressure is effective.

Coupled closely to the new wage equation is a price equation. A rise in prices leads to a rise in wages through the wage mechanism which in turn may lead to a further rise in prices; the familiar elements of *cost-push* inflation are thus built into the model. At this preliminary stage it is assumed that

$$\frac{1}{P}\frac{dP}{dt} = \S\left(\frac{1}{w}\frac{dw}{dt} - \alpha\right). \tag{3.23}$$

Prices are adjusted according to the extent that the rate of increase of money wages exceeds the real growth rate of output per man. For further information relating to Britain the reader is referred to the works of Dicks-Mireaux and Dow[14,15] where the interesting problems of parameter estimation are tackled. The extensive coupling between eqns. *(3.22)* and *(3.23)* calls for simultaneous estimation of the parameters using the technique of multi-stage regression.

The remainder of the basic model II is retained except for the inclusion of the price index to compensate for the measurement of goods and services which is in real terms. In particular the national accounting identity under the same classical savings assumptions leads to the equation

$$PY = W + PI$$

or $$I = \frac{dK}{dt} = \left(1 - \frac{w}{Pa}\right)Y. \tag{3.24}$$

The share of wages, u, equals $\frac{w}{Pa}$. Finally, as before, for employment

$$v = \frac{E}{L} = \frac{Y}{aL} \text{ and } Y = K/\sigma \tag{3.25}$$

and integration of the system can now proceed.

Logarithmic differentiation of the identity for share of wages gives

$$\frac{1}{u}\frac{du}{dt} = \frac{1}{w}\frac{dw}{dt} - \frac{1}{P}\frac{dP}{dt} - \alpha$$

and hence using the wage equation *(3.23)*

$$\xi\,\frac{1}{u}\frac{du}{dt} = (1 - \xi)\,\frac{1}{P}\frac{dP}{dt}$$

or, upon integration

$$\left.\begin{aligned} &P^{(1-\xi)} = Au^{\xi} \\ \text{or}\quad &P = A'u^{\delta} \end{aligned}\right\} \tag{3.26}$$

where $\delta = \xi/(1 - \xi)$ and also A' is constant. It would seem reasonable to assume that $\xi < 1$ (on the basis that wage costs are not the total component of product price) and hence that $\delta > 0$. Prices are linked directly to the share of wages. Of course if ξ is actually equal to unity the share of wages is fixed and the implication for prices remains to be investigated.

Now taking the wage equation *(3.22)* and substituting for w in terms of u,

$$\frac{1}{u}\frac{du}{dt} + \frac{1}{P}\frac{dP}{dt} + \alpha = -\gamma + \rho v + \eta\left(\frac{1}{P}\frac{dP}{dt} + \alpha\right)$$

i.e. $$\frac{1}{u}\frac{du}{dt} + (1 - \eta)\frac{1}{P}\frac{dP}{dt} = -\alpha(1 - \eta) - \gamma + \rho v$$

i.e. $$\frac{1}{u}\frac{du}{dt}\Big[1 + \delta(1 - \eta)\Big] = -\alpha(1 - \eta) - \gamma + \rho v. \tag{3.27}$$

The remaining equation in terms of u and v which completes the model is unaltered from eqn. *(3.6)* and is

$$\frac{1}{v}\frac{dv}{dt} = \frac{1}{\sigma}(1 - u) - (\alpha + \beta). \tag{3.28}$$

It is relevant to note that together eqns. *(3.27)* and *(3.28)* are identical in form to eqns. *(3.6)* which represent model II. The new solution may be deduced forthwith as

$$[1 + \delta(1 - \eta)]\{[\frac{1}{\sigma} + (\alpha + \beta)]\log u - \frac{1}{\sigma} u\}$$
$$= - [\alpha(1 - \eta) + \gamma] \log v + \rho v + constant. \qquad (3.29)$$

The implication of this result and its similarity to the previous solution of model II is that prices are very much peripheral to the basic structure of the economy under consideration, unless the effect of either of the newly introduced parameters is to critically change the stability. The new equilibrium is

$$\left.\begin{aligned} \bar{u} &= 1 - \sigma(\alpha + \beta) \\ \bar{v} &= [\alpha(1 - \eta) + \gamma]/\rho \end{aligned}\right\} \qquad (3.30)$$

and to examine stability again take small displacements δu and δv from this state. The system eqns. *(3.27)* and *(3.28)* reduce after linearisation to

$$\left.\begin{aligned} \delta\dot{u}[1 + \delta(1 - \eta)] &= \rho\delta v\bar{u} \\ \delta\dot{v} &= - \frac{1}{\sigma}\, \delta u\bar{v} \end{aligned}\right\} \qquad (3.31)$$

whence

$$\delta\ddot{u}[1 + \delta(1 - \eta)] = - \rho\bar{u}\, \frac{1}{\sigma}\, \bar{v}\delta u. \qquad (3.32)$$

The linearised approximation to motion is again simple harmonic provided that

$$1 + \delta(1 - \eta) > 0$$

i.e. $\eta < 1 + \frac{1}{\delta} = 1 + (1 - \xi)/\xi = \frac{1}{\xi}$

i.e. $\eta\xi < 1.$ *(3.33)*

When $\xi < 1$ the system will be unstable for sufficiently large $\eta > 1$, although the evidence (see the previous references[14,15,39] again) suggests that such high sensitivity of wages to value productivity is unlikely to be achieved.

In the special case of $\xi = 1$ however u is constant and eqn. *(3.28)* integrates to give an exponential solution for v. The system is then clearly unstable.

CHAPTER 4

NON - LINEAR DIFFERENTIAL - DELAY MODELS

The findings of the previous chapter have suggested that the price mechanism is merely an adjunct to the main structure and, as such, is not of itself likely to cause significant effects in behaviour. This is to be expected in the context of a closed model; prices are basically a numeraire for money magnitudes. Of course, for a nation with a substantial foreign trading account there will, as shown by Dicks-Mireaux[15] in the case of the U.K., be a clear dependence between prices on the home market and import prices, which leads to a further potentially interesting area of research in the analysis of multi-national models.

However from this point in the thesis we have elected to concentrate in more depth on aspects of investment and productivity rather than on prices. The models put forward in the current chapter improve on previous models in two significant respects. Firstly, time-delayed components are modelled explicitly and a truly vintage based productivity is included in the *vintage* model (Model III). The effect of population growth is investigated and found to be more significant than previously. Secondly the constant capital-output ratio constraint is relaxed in the *putty-clay* model (Model IV) which allows for a choice of technology. The resulting systems are expressed mathematically in terms of differential-delay equations (DDE).

The class of DDEs with which we are concerned is defined by

$$\frac{dx}{dt} = F(t, \underline{x}(*))$$

where $\underline{x}$ is an n vector whose argument is lagged (as denoted by an asterisk) and is moreover a function of the dependent variable, that is of the unknown solution itself. A simple example is the equation

$$\frac{dx}{dt} = x(t - x(t)),$$

in which x is a scalar variable dependent upon t.

Following the formulation of the *vintage* and *putty-clay* models and their reduction to DDE systems we investigate the existence of equilibrium states. An equilibrium state, $\underline{x}_e$ is a solution which satisfies the equation

$$F(t, \underline{x}_e) = 0$$

for all t. The determination of the equilibrium states does not involve the solution of the system differential equations, and the fact that lagged arguments are involved is at this point irrelevant. $\underline{x}_e$ corresponds to a constant solution of the system $(\dot{\underline{x}} = 0)$. Whilst non-linear systems can in general possess one or more equilibrium states, the equilibrium solutions found below are shown to be unique. Dynamic analysis of the models is the subject of later chapters.

4.1 The Basic Vintage Model (Model III)

According to standard practice in vintage model formulation (see Allen[1] for example) $Z(\tau)$ is taken as the aggregated output from all plant of vintage τ. Such plant employs $N(\tau)$ units of labour. Technical progress is embodied in the total capital stock which at any time is made up of machines of different vintages. There is no substitution between factors either before or after installation of a new machine, implying that once in operation a machine requires a fixed crew and that eventually the machine will be retired to release the crew for work on new and more productive plant. Again the economic life of plant, θ is of paramount significance.

As in sec. 3.2 θ is determined by the zero profit criterion. The rate of profit at time t on plant of vintage τ is

$$Z(\tau) - w(t)N(\tau).$$

This expression is zero for plant which is θ years old and hence

$$Z(t - \theta) = w(t)N(t - \theta). \tag{4.1}$$

Summing the output and employment on all productive plant leads to expressions for gross domestic output, Y and the total employment E at any time

$$Y(t) = \int_{t-\theta}^{t} Z(\tau)d\tau, \tag{4.2}$$

and

$$E(t) = \int_{t-\theta}^{t} N(\tau)d\tau. \tag{4.3}$$

The rate of growth of technical progress is taken as α (a constant) by assuming exponential growth in labour productivity on new plant in the form

$$Z(t) = a_0 N(t) \exp\{\alpha t\}. \tag{4.4}$$

The ratio Z/N can now be eliminated between eqns. *(4.1)* and *(4.4)* to give

$$Z(t - \theta)/N(t - \theta) = w(t) = a_0 \exp\{\alpha(t - \theta)\} \quad . \tag{4.5}$$

or $\quad \alpha(t - \theta) = \log\left(\frac{w}{a_0}\right),$

or $\quad \theta = \frac{1}{\alpha} \log\left(\frac{a_0}{w}\right) + t. \quad (4.6)$

This last equation compares directly with eqn. *(3.13)* of the previous chapter despite inherent differences in the definition of growth in labour productivity.

For the time being constancy of the capital-output ratio is retained and we have for gross investment, I

$$I(t)/Z(t) = \sigma, \text{ a constant.} \tag{4.7}$$

The models of the previous chapter are also extended by assuming a differential form of savings function in which s_W and s_P are the savings rates on wages and profits respectively. Savings are matched by current investment so that

$$I(t) = s_W w(t)E(t) + s_P[Y(t) - w(t)E(t)]. \tag{4.8}$$

The *classical savings* situation which may be of interest is the special case in which $s_W = 0$ and $0 < s_P \leq 1$.

Substitution for I from eqn. *(4.7)* and w from eqn. *(4.5)* in eqn. *(4.8)* leads to

$$\sigma Z = (s_W - s_P)a_0 \exp\{\alpha(t - \theta)\}E + s_P Y.$$

Substituting further for Y, E and Z from eqns. *(4.2)*, *(4.3)* and *(4.4)* respectively gives

$$\sigma a_0 N \exp\{\alpha t\} = (s_w - s_p)a_0 \exp\{\alpha(t - \theta)\}\int_{t-\theta}^{t} N(\tau)d\tau$$

$$+ s_p a_0 \int_{t-\theta}^{t} N(\tau) \exp\{\alpha\tau\}d\tau. \qquad (4.9)$$

The dependent variables in eqn. *(4.9)* are $N(t)$ and $\theta(t)$.

The model is completed by readopting the Phillips wage equation of model IIA, that is

$$\frac{1}{w}\frac{dw}{dt} = -\gamma + \frac{\rho L}{U},$$

or

$$\frac{1}{w}\frac{dw}{dt} = -\gamma + \frac{\rho}{1 - v} \qquad (4.10)$$

where $v = \int_{t-\theta}^{t} N(\tau)d\tau/L$.

Logarithmic differentiation of eqn. *(4.5)* permits the elimination of w giving

$$\alpha(1 - \frac{d\theta}{dt}) = -\gamma + \frac{\rho}{1 - v}. \qquad (4.11)$$

Eqns. *(4.9)* and *(4.11)* represent the final reduced form of the model.

Eqns. *(4.9)* and *(4.11)* yield a differential-delay system in that not only does N occur in the delayed form $N(t - \theta)$*, but also

* *This fact is perhaps more apparent when the differentiated form of eqn. (4.9) is considered (see eqns. (4.17) to follow).*

the time lag itself is a time dependent variable. Furthermore it will be seen later that the variability in θ is quite considerable thus invalidating attempts at small order approximation to fluctuations in the time lag. The model is of the *threshold* type referred to by Cooke[27] who emphasises two points which are *the potential importance of models of the threshold type and, second, the fact that such models lead to a variety of mathematical questions, few of which have so far been answered.* In fact it is both interesting and relevant to note that the above macro-economic model has much in common with some biological models of population and epidemic growth which have been discussed by Cooke and Yorke[13].

4.2 An Equilibrium Solution to the Basic Vintage Model

Labour supply is assumed constant for the *basic vintage model* and written as $L = L_0$, say. It is appropriate at this stage to introduce a non-dimensional variable

$$n(t) = N(t)/L = N/L_0, \tag{4.12}$$

whereupon eqn. *(4.9)* after division by a_0L_0 becomes

$$\sigma n \exp\{\alpha t\} = (s_w - s_p) \exp\{\alpha(t - \theta)\}\int_{t-\theta}^{t} n(\tau)d\tau$$
$$+ s_p\int_{t-\theta}^{t} n(\tau)\exp\{\alpha\tau\}d\tau. \tag{4.13}$$

The differentiated form of eqn. *(4.13)* is then

$$\sigma\left(\frac{dn}{dt} + \alpha n\right) \; exp\{\alpha t\} =$$

$$(s_w - s_p) \; exp\{\alpha(t - \theta)\}\alpha(1 - \frac{d\theta}{dt})\int_{t-\theta}^{t} n(\tau)d\tau$$

$$+ \; (s_w - s_p) \; exp\{\alpha(t - \theta)\}\left[n(t) - n(t - \theta)(1 - \frac{d\theta}{dt})\right]$$

$$+ \; s_p\left[n(t) \; exp\{\alpha t\} - n(t - \theta) \; exp\{\alpha(t - \theta)\}(1 - \frac{d\theta}{dt})\right].$$

Collecting terms and cancelling $exp\{\alpha t\}$ we have

$$\sigma \frac{dn}{dt} + \left[\alpha\sigma - s_p - (s_w - s_p) \; exp\{- \alpha\theta\}\right]n(t) =$$

$$(s_w - s_p) \; exp\{- \alpha\theta\}\alpha(1 - \frac{d\theta}{dt})\int_{t-\theta}^{t} n(\tau)d\tau$$

$$- \; s_w \; exp\{- \alpha\theta\}(1 - \frac{d\theta}{dt})n(t - \theta). \tag{4.14}$$

Finally we have

$$v = \int_{t-\theta}^{t} N(\tau)d\tau/L_0 = \int_{t-\theta}^{t} n(\tau)d\tau \tag{4.15}$$

or in its differentiated form

$$\frac{dv}{dt} = n(t) - n(t - \theta)(1 - \frac{d\theta}{dt}). \tag{4.16}$$

Writing $n \equiv n(t)$ and $n^* \equiv n(t - \theta)$ and substituting v in eqn. *(4.14)* the model may be described by the following system of ordinary differential-delay equations

$$\left.\begin{aligned}
&\sigma\frac{dn}{dt} + \left[\alpha\sigma - s_p - (s_w - s_p)\ exp\{-\ \alpha\theta\}\right]n = \\
&\quad exp\{-\ \alpha\theta\}\left[(s_w - s_p)\alpha v - s_w n^*\right](1 - \frac{d\theta}{dt}), \\
&\frac{dv}{dt} = n - n^*(1 - \frac{d\theta}{dt}), \\
&\alpha\frac{d\theta}{dt} = (\alpha + \gamma) - \frac{\rho}{1 - v}.
\end{aligned}\right\} \quad (4.17)$$

The solution to eqns. *(4.17)* will coincide with the solution to the original model for some appropriately chosen set of arbitrary constants. In other words the process of differentiation has caused a loss of certain information.

To investigate the stationary equilibrium set $\dot{n} = \dot{v} = \dot{\theta} = 0$ and $n^* = n$ in the usual way. Denoting equilibrium values by $\bar{n}$, $\bar{v}$ and $\bar{\theta}$ we have

$$\left[(\alpha\sigma - s_p)\ exp\{\alpha\bar{\theta}\} - (s_w - s_p)\right]\bar{n} = (s_w - s_p)\alpha\bar{v} - s_w\bar{n}, \quad (4.18)$$

and $$\bar{v} = 1 - \frac{\rho}{\alpha + \gamma} \quad (4.19)$$

from the first and third of eqns. *(4.17)*. The second of these equations does not yield a solution, but reverting to the integrated form in eqn. *(4.15)* one obtains

$$\bar{v} = \int_{t-\bar{\theta}}^{t} \bar{n}d\tau = \bar{n}\bar{\theta}. \quad (4.20)$$

Substituting for $\bar{v}$ in eqn. *(4.18)* and cancelling $\bar{n}$,

$$(s_p - \alpha\sigma)\ exp\{\alpha\bar{\theta}\} = s_p + (s_p - s_w)\alpha\bar{\theta}. \quad (4.21)$$

Eqn. *(4.21)* yields a unique solution for $\bar{\theta}$. Subsequent substitution in eqns. *(4.19)* and *(4.20)* gives the corresponding values for $\bar{v}$ and $\bar{n}$. It is assumed that $1 > s_p > s_w \geq 0$ and also that $s_p > \alpha\sigma$, so that

$$f_1(\theta) \equiv (s_p - \alpha\sigma)\ exp\{\alpha\theta\}$$

is an exponentially increasing function of θ, and

$$f_2(\theta) \equiv s_p + (s_p - s_w)\alpha\theta$$

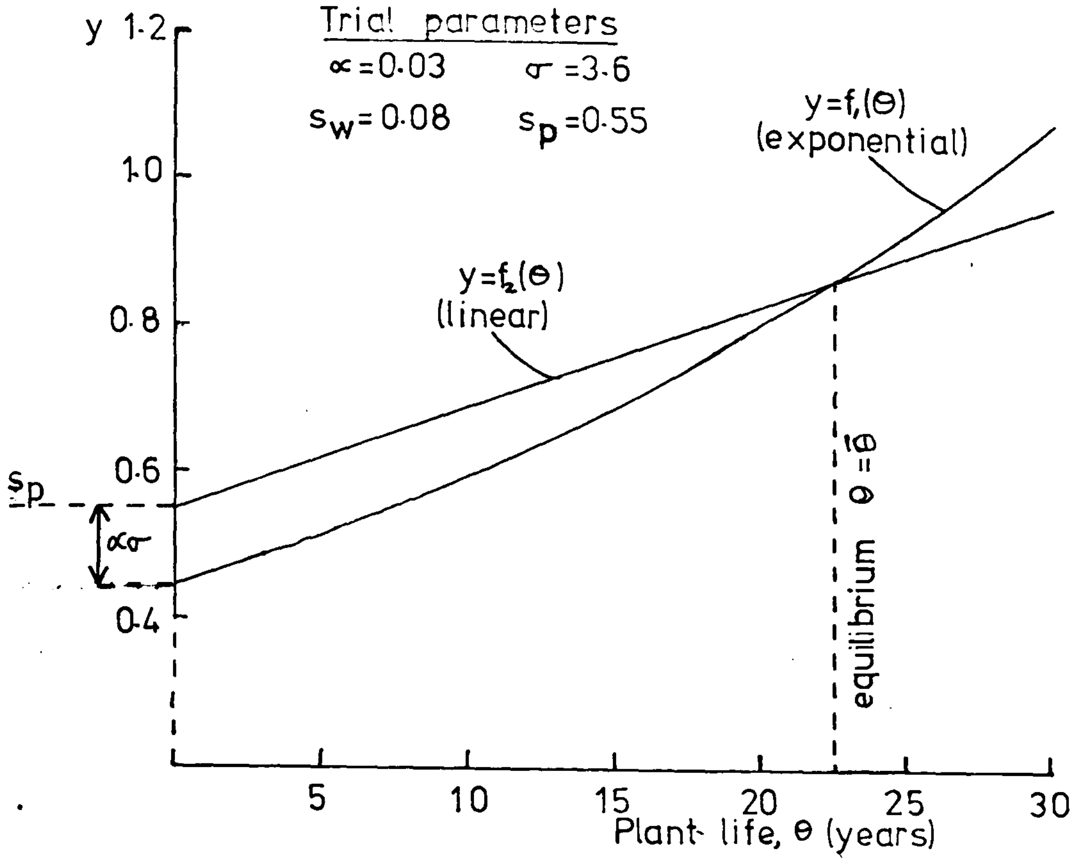
is a linearly increasing function of θ. Furthermore we have that

$$f_1(0) = s_p - \alpha\sigma < s_p = f_2(0),$$

which guarantees a unique intersection to the curves $y = f_1(\theta)$ and $y = f_2(\theta)$ for some positive $\theta = \bar{\theta}$ as demonstrated in fig. 8*. The equilibrium life of plant is seen to vary between about 15 and 25 years depending upon the parameter values chosen. The equilibrium value of employment v has already been seen to be independent of plant life and its value is clearly very dependent on the parameters in the wage equation. In equilibrium wages and output are of course growing, both at the constant rate α (this is evident from eqns. *(4.4) and (4.5)*) and the economy is said to be in a state of *balanced growth.*

Implicit in the present model has been the positivity of α. Whilst growth in technical progress cannot by definition be negative, the limiting case of zero growth is of some interest and is mentioned briefly in the next section.

* *Eqn. (4.21) also has a root on the negative θ axis but this does not represent a meaningful solution in terms of economics.*



(a) Uniqueness of solution

s_w	s_p	α	$\bar{\theta}$ (years)
0	1	2%	22.0
		3%	18.5
		5%	15.5
0.08	0.55	2%	24.7
		3%	22.5
		5%	20.4

(b) Typical solution values ($\sigma = 3.6$)

FIGURE 8 Equilibrium solution for basic vintage model – determination of value for plant life.

4.3 The Special Case of Zero Growth in the Vintage Model

For zero growth, $\alpha = 0$, all plant is equally productive and must therefore be assumed always operational. In other words the lower limits in eqns. *(4.2)* and *(4.3)* revert to minus infinity. Whilst this interpretation is not strictly the limiting case of the *vintage model* because the zero profit criterion has been abandoned, it is the only meaningful alternative. Additionally, with zero growth, the national average productivity Y/E is equal to the marginal productivity on new investment Z/N.

Under these circumstances the model is identical, apart from the Phillips curve modification, to Model II with $\alpha = 0$.

4.4 The Vintage Model with Population Growth (Model IIIA)

It is clearly pertinent in growth modelling to understand the effects of variation in labour supply. Whilst the concept of an expanding economy with a growing labour force may seem a happily realisable situation, is the advent of the shorter working week, longer holidays, more education, etc. perhaps linked with a stabilisation in total population likely to jeopardise subsequent events? In the present section we assume labour supply, L to equal $L_0\ exp\{\beta t\}$ and consider both positive and negative values of β.

In the modified model relative employment becomes

$$v = \int_{t-\theta}^{t} N(\tau)d\tau \;/\; (L_0 \; exp\{\beta t\})$$

$$= exp\{-\;\beta t\}\int_{t-\theta}^{t} n(\tau)\; exp\{\beta\tau\}d\tau \qquad . \qquad (4.22)$$

where the non-dimensional variable n is now defined by

$$n(t) \;=\; N(t) \;/\; (L_0 \; exp\{\beta t\}). \qquad (4.23)$$

(One must expect that in equilibrium growth N will grow at a rate β to compensate for the growth in labour supply. This has conditioned the above choice of n; equilibrium solutions can again be sought by setting $\dot{n} = 0$.) The differentiated form of the above relationship which replaces eqn. *(4.16)* is

$$\frac{dv}{dt} = -\;\beta v + n(t) \;-\; exp\{-\;\beta\theta\}n(t\;-\;\theta)(1\;-\;\frac{d\theta}{dt}). \qquad (4.24)$$

In amending eqn. *(4.14)* n is replaced by $n\; exp\{\beta t\}$ which gives, after division by $exp\{\beta t\}$,

$$\sigma(\frac{dn}{dt} + \beta n) \;+\; [\alpha\sigma \;-\; s_p \;-\; (s_w \;-\; s_p)\; exp\{-\;\alpha\theta\}]n(t) \;=$$

$$(s_w \;-\; s_p)\; exp\{-\;\alpha\theta \;-\; \beta t\}\alpha(1\;-\;\frac{d\theta}{dt})\int_{t-\theta}^{t} n(\tau)\; exp\{\beta\tau\}d\tau$$

$$-\; s_w\; exp\{-\;(\alpha\;+\;\beta)\theta\}(1\;-\;\frac{d\theta}{dt})n(t\;-\;\theta). \qquad (4.25)$$

Substitution of v in place of the integral gives the final modified form of

$$\sigma\;\frac{dn}{dt} + [(\alpha\;+\;\beta)\sigma \;-\; s_p \;-\; (s_w \;-\; s_p)\; exp\{-\;\alpha\theta\}]n \;=$$

$$exp\{-\;\alpha\theta\}[(s_w \;-\; s_p)\alpha v \;-\; s_w\; exp\{-\;\beta\theta\}n^*](1\;-\;\frac{d\theta}{dt}). \qquad (4.26)$$

Together with the unmodified equation

$$\alpha \frac{d\theta}{dt} = (\alpha + \gamma) - \frac{\rho}{1 - v}, \tag{4.11}$$

eqns. *(4.24)* and *(4.26)* define the system.

For a stationary equilibrium in n, v and θ we have

$$\bar{v} = 1 - \frac{\rho}{\alpha + \gamma} \tag{4.19}$$

as before. From eqn. *(4.24)*

$$\beta\bar{v} = \bar{n}(1 - exp\{- \beta\bar{\theta}\}), \tag{4.27}$$

and from eqn. *(4.26)*

$$[(\alpha + \beta)\sigma - s_p]\ exp\{\alpha\bar{\theta}\} - (s_w - s_p) =$$

$$(s_w - s_p)\ \frac{\alpha}{\beta}\ (1 - exp\{- \beta\bar{\theta}\}) - s_w\ exp\{- \beta\theta\},$$

or equivalently

$$[s_p - (\alpha + \beta)\sigma]\ exp\{\alpha\bar{\theta}\} =$$

$$(s_p - s_w)(1 + \frac{\alpha}{\beta})(1 - exp\{- \beta\bar{\theta}\}) + s_p\ exp\{- \beta\bar{\theta}\}. \tag{4.28}$$

Once again if a solution for $\bar{\theta}$ can be found from eqn. *(4.28)*, $\bar{n}$ is determined by eqn. *(4.27)* given $\bar{v}$ from eqn. *(4.19)*. Let

$$f_1(\theta) = [s_p - (\alpha + \beta)\sigma]\ exp\{\alpha\theta\}$$

which is an exponentially increasing function of θ assuming now that $s_p > (\alpha + \beta)\sigma$, and let

$$f_2(\theta) = (s_p - s_w)(1 + \frac{\alpha}{\beta})(1 - exp\{- \beta\theta\}) + s_p\ exp\{- \beta\theta\}.$$

To examine possible solutions to eqn. *(4.28)* by considering the intersection of the curves $y = f_1(\theta)$ and $y = f_2(\theta)$ several separate situations are identified. In each case however note that $f_1(0) = s_p - (\alpha + \beta)\sigma$ and $f_2(0) = s_p$.

4.4.1 Case $\beta > \alpha\left(\dfrac{s_p - s_w}{s_w}\right) > 0$

(only applicable if $s_w > 0$)

$f_1(\theta)$ is an exponentially increasing function of θ and $f_1(0) < f_2(0)$. $f_2(\theta)$ exhibits exponential decay to a limiting value given by $f_2(\infty) = (s_p - s_w)(1 + \frac{\alpha}{\beta})$. For the assumed condition on β $f_2(\infty) < f_2(0)$, and the solution of the equation $f_1(\theta) = f_2(\theta)$ is clearly unique as shown in fig. 9(a).

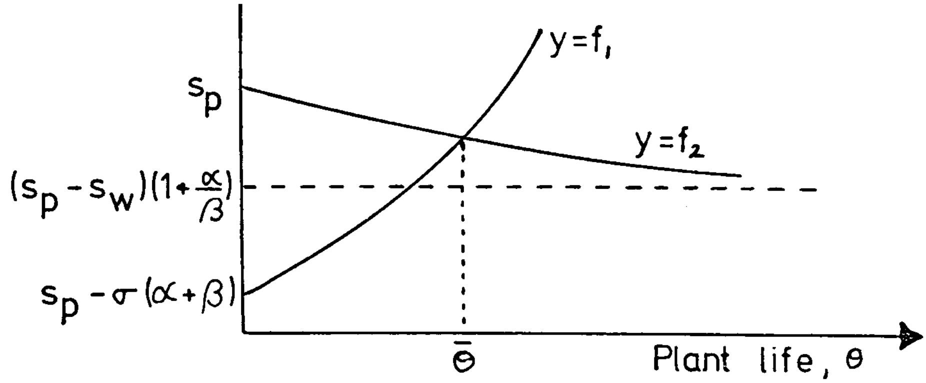
4.4.2 Case $\alpha\left(\dfrac{s_p - s_w}{s_w}\right) > \beta > 0$

(or all $\beta > 0$ if $s_w = 0$)

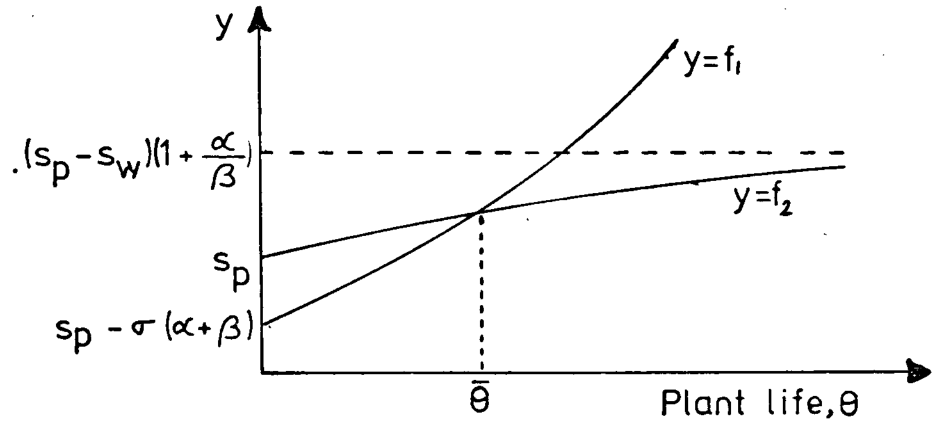
This case is as for the previous case except that $f_2(\infty) > f_2(0)$. The function $f_2(\theta)$ approaches a limiting value from below as in fig. 9(b).

4.4.3 Case $0 > \beta > -\alpha$

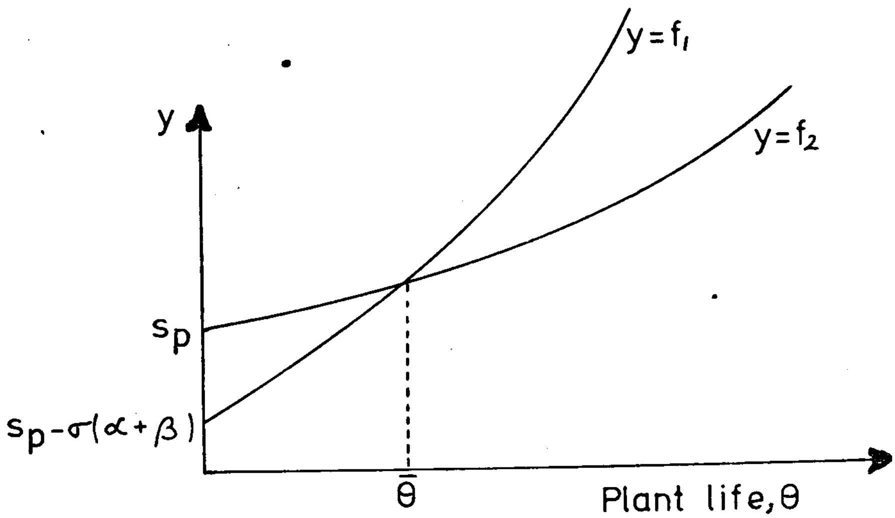
With negative β both $f_1(\theta)$ and $f_2(\theta)$ are exponentially increasing functions. For the assumed condition $f_1(0) < f_2(0)$, but the growth rate of f_1 is greater than that of f_2 so that an intersection is again guaranteed. See fig. 9(c).



(a) $\beta > \alpha\left(\frac{s_p - s_w}{s_w}\right) > 0$



(b) $\alpha\left(\frac{s_p - s_w}{s_w}\right) > \beta > 0$



(c) $0 > \beta > -\alpha$

FIGURE 9. Equilibrium solution for vintage model with population

4.4.4 Case $\alpha + \beta = 0$

This case calls for special attention in view of the fact that the functions $f_1(\theta)$ and $f_2(\theta)$ are identical, thus apparently admitting any solution for θ. Whilst the operating condition $\alpha + \beta = 0$ is probably unlikely for any present day advanced economy, it may be worth pursuing in the sense of a limiting case.

Let $\beta = \varepsilon - \alpha$ and consider the limit as $\varepsilon \to 0$. Returning to eqn. *(4.28)* whose solution gives the equilibrium value(s) of θ, we have

$$(s_p - \varepsilon\sigma)\ exp\{\alpha\bar{\theta}\} =$$

$$(s_p - s_w)\left(\frac{\varepsilon}{\varepsilon - \alpha}\right)(1 - exp\{(\alpha - \varepsilon)\bar{\theta}\}) + s_p\ exp\{(\alpha - \varepsilon)\bar{\theta}\}.$$

Ignoring second order small quantities

$$(s_p - \varepsilon\sigma)\ exp\{\alpha\bar{\theta}\} =$$

$$(s_p - s_w)\ \frac{\varepsilon}{-\alpha}\ (1 - exp\{\alpha\bar{\theta}\}) + s_p\ exp\{\alpha\bar{\theta}\}(1 - \varepsilon\bar{\theta}),$$

that is

$$\alpha(s_p\bar{\theta} - \sigma) = (s_p - s_w)(1 - exp\{-\alpha\bar{\theta}\}). \qquad (4.29)$$

Once again the nature of eqn. *(4.29)* is such as to determine a unique positive (non-zero) limiting value for $\bar{\theta}$ as $\alpha + \beta \to 0$.

This result serves to indicate the ill-conditioned nature of the equilibrium system as $\alpha + \beta$ approaches zero. The limiting case is of particular interest to the *world modeller* who is worried about resource depletion (for instance see the original work of Forrester[21]), for it represents zero growth in GNP. The growth rate of productivity, α is exactly matched by the decline in man-hours worked, β to stabilise consumption. However it is probably safe to assume that such a state would be hazardous to operate in view of the problems inherent in an ill-conditioned system.

Operating states in which $\alpha + \beta < 0$ are not considered relevant at the present implying as they do a reduction in standard of living (at least as measured by economic factors).

Before turning to the dynamic properties of the above models a further formulation is introduced in which the constant capital-output ratio condition is relaxed.

4.5 The Putty-Clay Model (Model IV)

The previous constant capital-output ratio models have implied a fixed capital-labour ratio at the moment that investment occurs without the possibility of substitution between factors. In contrast Bliss[7] has undertaken the rigorous analysis of a generalised model of economic growth which is characterised by a variable capital-labour ratio at the time of investment (*putty*), and a fixed capital-labour ratio thereafter (*clay*). The type of investment is chosen according to a profit maximising criterion (which naturally requires judgement on the future course of events). The extensive analysis of Bliss examines the state of balanced growth in a situation of full employment, but in building the *putty-clay* assumptions into the present *disequilibrium* models one might hopefully seek similar results to Bliss in terms of the existence and uniqueness of equilibrium solutions.

This expansion of the investment sector entails the definition of the following new variables:

$k(t) = I/N$,	per capita investment on new plant
$V(t)$	the present value of returns on an investment made at time t, that is an investment in vintage t plant
$r(t)$	the rate of return on investments made at time t
$\phi(t)$	the economic life of new plant as anticipated by entrepreneurs at time t
$\lambda(t)$	the growth rate of wages as anticipated by entrepreneurs at time t.

Eqns. *(4.1)*, *(4.2)*, *(4.3)*, *(4.8)* and *(4.10)* from the previous model are retained. To complete the production and employment sector a Cobb-Douglas production function of the type

$$\sigma Z = I^b N^{1-b} \exp\{\alpha(1 - b)t\} \tag{4.30}$$

is adopted where α and b are positive constants and b is also less than unity. (Whilst it is tempting to interpret the technical progress as solely labour augmented this is an unnecessarily restrictive view - the factor of $(1 - b)$ in the exponential term is merely a device to simplify subsequent algebraic reduction.) The choice of this particular production function with constant returns to scale represents a compromise between the much broader class of production function considered by Bliss and the need to retain simplicity so that the resulting algebra is manageable.

In the investment sector entrepreneurs make a profit maximising decision, the success of which is based on their ability to correctly predict the future time path of wage rates. This predicted time path is taken as exponential at a rate λ (that is an expectation of constant growth at any particular instant of time although the value of that constant will be reviewed from time to time in the light of experience). The future profit rate on new investment is thus anticipated as

$$Z(t) - w(t)\ exp\{\lambda\tau\}N(t)$$

after τ years. The estimate of ϕ is thus given according to the zero profit criterion by

$$Z(t) = w(t)\ exp\{\lambda\phi\}N(t). \tag{4.13}$$

The present value of returns anticipated as a result of current investment, $I(t)$ is given by

$$V = \int_t^{t+\phi} [Z(t) - N(t)w(t)\ exp\{\lambda(\tau - t)\}]\ exp\{-\ r(\tau - t)\}d\tau, \tag{4.32}$$

where $r(t)$ is the current rate of return or discount rate which is constant for all plant of one vintage but time dependent in the same sense as λ. To follow Bliss, the capital intensity of each type of investment is chosen so as to maximise the net present value of an investment designed to employ one man, and this net present value is zero by suitable choice of discount rate r; there is no pure profit.

$$V - I = 0 \quad \text{subject to} \ \frac{\partial}{\partial k}\left(\frac{V}{N} - k\right) = 0 \tag{4.33}$$

at any instant of time t. Strictly, of course, eqn. *(4.33)* does not in general guarantee a maximum but for the particular production function specified in eqn. *(4.30)* it proves to be sufficient.

Alternatively it is possible to define r as that rate of return which determines that there shall be no pure profit and then choose the capital intensity so as to maximise r. Eqn. *(4.33)* is then replaced by

$$\frac{\partial r}{\partial k} = 0 \quad \text{subject to } V - I = 0 \tag{4.33a}$$

at all instants of time t. In the next section it will be shown that eqns. *(4.33)* and *(4.33a)* are in fact equivalent in terms of the present model.

4.6 Reduction of the Putty-Clay Model

A change of variable, $s = \tau - t$, in the integral of eqn. *(4.32)* gives

$$V = \int_0^{\phi} [Z - Nw \exp\{\lambda s\}] \exp\{-rs\}ds$$

$$= Z\int_0^{\phi} \exp\{-rs\}ds - Nw \int_0^{\phi} \exp\{-(r - \lambda)s\}ds$$

$$= Z\,\frac{1}{r}\,(1 - \exp\{-r\phi\}) - Nw\,\frac{1}{r - \lambda}\,(1 - \exp\{-(r - \lambda)\phi\}). \tag{4.34}$$

But from eqn. *(4.30)*

$$Z = \frac{1}{\sigma}\,I^b N^{1-b} \exp\{\alpha(1 - b)t\} = \frac{1}{\sigma}\,k^b N \exp\{\alpha(1 - b)t\}$$

and from eqn. *(4.31)*

$$Nw = Z \exp\{-\lambda\phi\}.$$

Hence substitution in eqn. *(4.34)* gives

$$V/N = \frac{1}{\sigma} k^b \exp\{\alpha(1 - b)t\}\left[\frac{1}{r}(1 - \exp\{-r\phi\}) - \frac{1}{r - \lambda}(\exp\{-\lambda\phi\} - \exp\{-r\phi\})\right]. \quad (4.35)$$

Furthermore, eliminating w between eqns. *(4.1)* and *(4.31)* gives

$$w = \frac{Z(t - \theta)}{N(t - \theta)} = \exp\{-\lambda\phi\} \frac{Z(t)}{N(t)},$$

that is

$$\frac{1}{\sigma} k^b(t - \theta) \exp\{\alpha(1 - b)(t - \theta)\} =$$

$$\exp\{-\lambda\phi\} \frac{1}{\sigma} k^b(t) \exp\{\alpha(1 - b)t\},$$

or

$$\exp\{\lambda\phi\} = \frac{k^b(t)}{k^b(t - \theta)} \exp\{\alpha(1 - b)\theta\}, \quad (4.36)$$

or

$$\lambda\phi = b[\log k(t) - \log k(t - \theta)] + \alpha(1 - b)\theta$$

from which

$$\lambda \frac{\partial\phi}{\partial k}_{(t\ \text{constant})} = \frac{b}{k}. \quad (4.37)$$

To follow Bliss differentiate eqn. *(4.35)* partially with respect to k with t constant,

$$\frac{\partial}{\partial k}\left(\frac{V}{N}\right) = \frac{b}{k}\left(\frac{V}{N}\right) + \frac{1}{\sigma} k^b \exp\{\alpha(1 - b)t\}\left[\exp\{-r\phi\} - \frac{1}{r - \lambda}(-\lambda\exp\{-\lambda\phi\} + r\exp\{-r\phi\})\right]\frac{\partial\phi}{\partial k}$$

$$= \frac{b}{k}\left(\frac{V}{N}\right) + \frac{1}{\sigma}\frac{1}{r - \lambda} k^b \exp\{\alpha(1 - b)t\} \times (\exp\{-\lambda\phi\} - \exp\{-r\phi\})\frac{b}{k}.$$

But from eqn. *(4.33)*

$$(V - I)/N = V/N - k = 0$$

subject to $\frac{\partial}{\partial k}\left(\frac{V}{N} - k\right) = \frac{\partial}{\partial k}\left(\frac{V}{N}\right) - 1 = 0,$

and thus

$$\frac{\partial}{\partial K}\left(\frac{V}{N}\right) = b + \frac{b}{\sigma}\,\frac{1}{r - \lambda}\,k^{b-1}\,exp\{\alpha(1 - b)t\}(exp\{-\lambda\phi\} - exp\{-r\phi\}) = 1,$$

or rearranging

$$\frac{b}{r - \lambda}\,(exp\{-\lambda\phi\} - exp\{-r\phi\}) = \sigma(1 - b)k^{1-b}\quad exp\{-\alpha(1 - b)t\}. \tag{4.38}$$

In association with eqn. *(4.38)* $V/N = k$ which by rearrangement of eqn. *(4.35)* leads to

$$\frac{1}{r}\,(1 - exp\{-r\phi\}) - \frac{1}{r - \lambda}\,(exp\{-\lambda\phi\} - exp\{-r\phi\}) =$$
$$\sigma k^{1-b}\,exp\{-\alpha(1 - b)t\}. \tag{4.39}$$

Eqns. *(4.38)* and *(4.39)* together effectively determine the choice of investment. For given λ and ϕ the per capita investment, k and the rate of return, r are determined simultaneously.

Before finally completing the reduction it is interesting to reflect on the alternative investment objective represented by eqn. *(4.33a)*. We first set $V = I$ which gives eqn. *(4.39)* and then differentiate this relation partially with respect to k for each instant of time where both r and ϕ are functions of k. However r is maximised by setting $\frac{\partial r}{\partial k} = 0$ giving precisely that same differential expression which led to eqn. *(4.38)*. The investment behaviours characterized by eqns. *(4.33)* and *(4.33a)* are entirely equivalent.

This result is quite general in the sense that it is not a feature of any particular production function for instance, nor is it solely true of equilibrium or balanced growth paths.

Reverting to the reduction of the model, substitute for the various variables in eqn. *(4.8)* to give

$$kN = s_p\int_{t-\theta}^{t} \frac{1}{\sigma}\, k^b(\tau)N(\tau)\ \exp\{\alpha(1 - b)\tau\}d\tau$$

$$- (s_p - s_w)\ \frac{1}{\sigma}\, k^b(t - \theta)\ \exp\{\alpha(1 - b)(t - \theta)\}\int_{t-\theta}^{t} N(\tau)d\tau, \tag{4.40}$$

and in eqn. *(4.10)*

$$\begin{aligned}
\frac{1}{w}\frac{dw}{dt} &= \frac{d}{dt}\log w \\
&= \frac{d}{dt}\log\left[\frac{1}{\sigma}\, k^b(t - \theta)\ \exp\{\alpha(1 - b)(t - \theta)\}\right] \\
&= \frac{d}{dt}\Big[b\ \log k(t - \theta) + \alpha(1 - b)(t - \theta)\Big] \\
&= \frac{d}{d(t - \theta)}\ [b\ \log k(t - \theta) + \alpha(1 - b)(t - \theta)](1 - \frac{d\theta}{dt}) \\
&= \left[b\ \frac{\dot{k}(t - \theta)}{k(t - \theta)} + \alpha(1 - b)\right](1 - \frac{d\theta}{dt}) \\
&= -\gamma + \frac{\rho}{1 - v}.
\end{aligned} \tag{4.41}$$

Finally substituting $n = N/L$ and $c = k\ \exp\{-\alpha t\}$ the model is reduced to the following system of mixed integro-differential equations in the main dependent variables n and θ

$$\left.\begin{aligned} \sigma cn &= s_p \exp\{-\alpha t\}\int_{t-\theta}^{t} c^b(\tau)n(\tau)\exp\{\alpha\tau\}d\tau \\ &\qquad - (s_p - s_w)c^b(t-\theta)\exp\{-\alpha\theta\}\int_{t-\theta}^{t} n(\tau)d\tau, \\ \left[b\,\frac{\dot{c}(t-\theta)}{c(t-\theta)} + \alpha\right]&\left(1 - \frac{d\theta}{dt}\right) = -\gamma + \rho/\left(1 - \int_{t-\theta}^{t} n(\tau)d\tau\right). \end{aligned}\right\} \quad (4.42)$$

where $c(t)$ is given by the algebraic equations

$$\left.\begin{aligned} \sigma c^{1-b} &= \frac{1}{r}(1 - \exp\{-r\phi\}) - \frac{1}{r-\lambda}(\exp\{-\lambda\phi\} - \exp\{-r\phi\}) \\ &= \frac{b}{r}(1 - \exp\{-r\phi\}). \end{aligned}\right\} \quad (4.43)$$

The double eqn. *(4.43)* in c and r is derived from an elementary transformation of eqns. *(4.38)* and *(4.39)*, and ϕ is given by eqn. *(4.36)* which after substitution for k becomes

$$\exp\{\lambda\phi\} = \frac{c^b(t)}{c^b(t-\theta)}\exp\{\alpha\theta\}. \quad (4.44)$$

λ it will be remembered is the future growth rate of wages as anticipated by entrepreneurs, and in the present instance it is assumed that wages are expected to continue growing at their current rate so that

$$\lambda = \frac{1}{w}\frac{dw}{dt} = -\gamma + \rho/\left(1 - \int_{t-\theta}^{t} n(\tau)d\tau\right). \quad (4.45)$$

Thus the model is now completely determined. Eqns. *(4.42)* possess exactly those characteristics of the vintage model noted earlier, but the complexity is compounded by the addition of the algebraic eqns. *(4.43)* through *(4.45)*.

4.7 An Equilibrium Solution to the Putty-Clay Model

Proceeding in the usual manner we set $c(t) = c(t - \theta) = \bar{c}$, $n(t) = n(t - \theta) = \bar{n}$ and $\theta(t) = \bar{\theta}$. (Once again the choice of variables is significant; a stationary condition on c implies the search for a balanced growth equilibrium in which per capita investment, k is growing at the exponential rate α in tune with GNP.) Substituting the equilibrium conditions directly in eqns. *(4.42)* gives

$$\left.\begin{aligned} \sigma\bar{c}\bar{n} &= s_p\, exp\{-\alpha t\}\int_{t-\bar{\theta}}^{t} \bar{c}^b\bar{n}\, exp\{\alpha\tau\}d\tau \\ &\quad - (s_p - s_w)\bar{c}^b\, exp\{-\alpha\bar{\theta}\}\bar{n}\bar{\theta} \\ \alpha &= -\gamma + \rho/(1 - \int_{t-\bar{\theta}}^{t} \bar{n}d\tau) \end{aligned}\right\}$$

that is

$$\left.\begin{aligned} \sigma\bar{c}^{1-b} &= \frac{s_p}{\alpha}(1 - exp\{-\alpha\bar{\theta}\}) - (s_p - s_w)\, exp\{-\alpha\bar{\theta}\}\bar{\theta}, \\ \bar{n}\bar{\theta}\ (= \bar{v}) &= 1 - \frac{\rho}{\alpha + \gamma}. \end{aligned}\right\} \tag{4.46}$$

Furthermore eqn. *(4.44)* gives

$$\lambda\bar{\phi} = \alpha\bar{\theta},$$

where $\lambda = -\gamma + \dfrac{\rho}{(1 - \bar{v})} = \alpha$

and thus $\bar{\phi} = \bar{\theta}$ (this is effectively a check; entrepreneurs anticipate that the life of new plant, $\bar{\phi}$ will be the same as it has always been in the past, namely $\bar{\theta}$). Substituting in eqns. *(4.43)*

$$\left.\begin{aligned}\sigma\bar{c}^{1-b} &= \frac{1}{\bar{r}}\,(1 - \exp\{-\bar{r}\bar{\theta}\}) - \frac{1}{\bar{r}-\alpha}\,(\exp\{-\alpha\bar{\theta}\} - \exp\{-\bar{r}\bar{\theta}\}) \\ &= \frac{b}{\bar{r}}\,(1 - \exp\{-\bar{r}\bar{\theta}\}), \\ &= \frac{s_p}{\alpha}\,(1 - \exp\{-\alpha\bar{\theta}\}) - (s_p - s_w)\,\exp\{-\alpha\bar{\theta}\}\bar{\theta}\end{aligned}\right\} \quad (4.47)$$

by use of eqn. *(4.46)*. The right hand sides of eqn. *(4.47)* may then be solved for $\bar{r}$ and $\bar{\theta}$ whence $\bar{c}$, $\bar{n}$ and $\bar{v}$ are determined by eqn. *(4.46)*. A particularly simple solution to eqns. *(4.47)* in which $\bar{r} = \alpha$ is available in the special case considered next.

4.7.1 Case $(1 - b)s_w = b(1 - s_p)$

Now

$$\lim_{\bar{r}\to\alpha} \frac{1}{\bar{r}-\alpha}\,(\exp\{-\alpha\bar{\theta}\} - \exp\{-\bar{r}\bar{\theta}\})$$

$$= \exp\{-\alpha\bar{\theta}\} \lim_{\bar{r}\to\alpha} \frac{1}{\bar{r}-\alpha}\,(1 - \exp\{-(\bar{r}-\alpha)\bar{\theta}\})$$

$$= \exp\{-\alpha\bar{\theta}\}\bar{\theta}.$$

$\bar{r} = \alpha$ is therefore a solution to eqns. *(4.47)* if and only if the following equations are consistent

$$\left.\begin{aligned}&\frac{1}{\alpha}\,(1 - \exp\{-\alpha\bar{\theta}\}) - \exp\{-\alpha\bar{\theta}\}\bar{\theta} \\ &= \frac{b}{\alpha}\,(1 - \exp\{-\alpha\bar{\theta}\}) \\ &= \frac{s_p}{\alpha}\,(1 - \exp\{-\alpha\bar{\theta}\}) - (s_p - s_w)\,\exp\{-\alpha\bar{\theta}\}\bar{\theta}.\end{aligned}\right\} \quad (4.48)$$

The condition for consistency is

$$\begin{vmatrix} (\frac{1}{\alpha} - \frac{b}{\alpha}) & -1 \\ (\frac{s_p}{\alpha} - \frac{b}{\alpha}) & -(s_p - s_w) \end{vmatrix} = 0,$$

that is

$$(1 - b)(s_p - s_w) - (s_p - b) = 0$$

or $$(1 - b)s_w = b(1 - s_p). \tag{4.49}$$

One reason why this solution is of particular interest is that *classical savings* in which $s_w = 0$ and $s_p = 1$, is one such special case whatever the value of b.

If the condition of eqn. *(4.49)* is satisfied the equilibrium solution is given according to eqn. *(4.48)* by

$$(1 - b)(1 - exp\{-\alpha\bar{\theta}\}) = exp\{-\alpha\bar{\theta}\}\alpha\bar{\theta}$$

or $$exp\{\alpha\bar{\theta}\} = 1 + \frac{\alpha\bar{\theta}}{1 - b}. \tag{4.50}$$

Eqn. *(4.50)* will clearly posses a unique positive solution (discounting the zero solution) whose value will depend solely on the parameters α and b. Fig. 10 depicts the equilibrium solution for $\bar{\theta}$ versus b for various values of α. For instance with α = 3% and b = 0·25 the equilibrium life of plant is about 18 years.

4.7.2 Case $(1 - b)s_W \neq b(1 - s_P)$

The non-linear nature of eqns. *(4.47)* makes it difficult to determine equilibrium in the general case except by resort to numeric computation. Program ES4 in appendix 2 is an example of how the latter may be achieved.

It is perhaps worth noting however that if the non-equality which represents the present case is only marginal, so that solutions

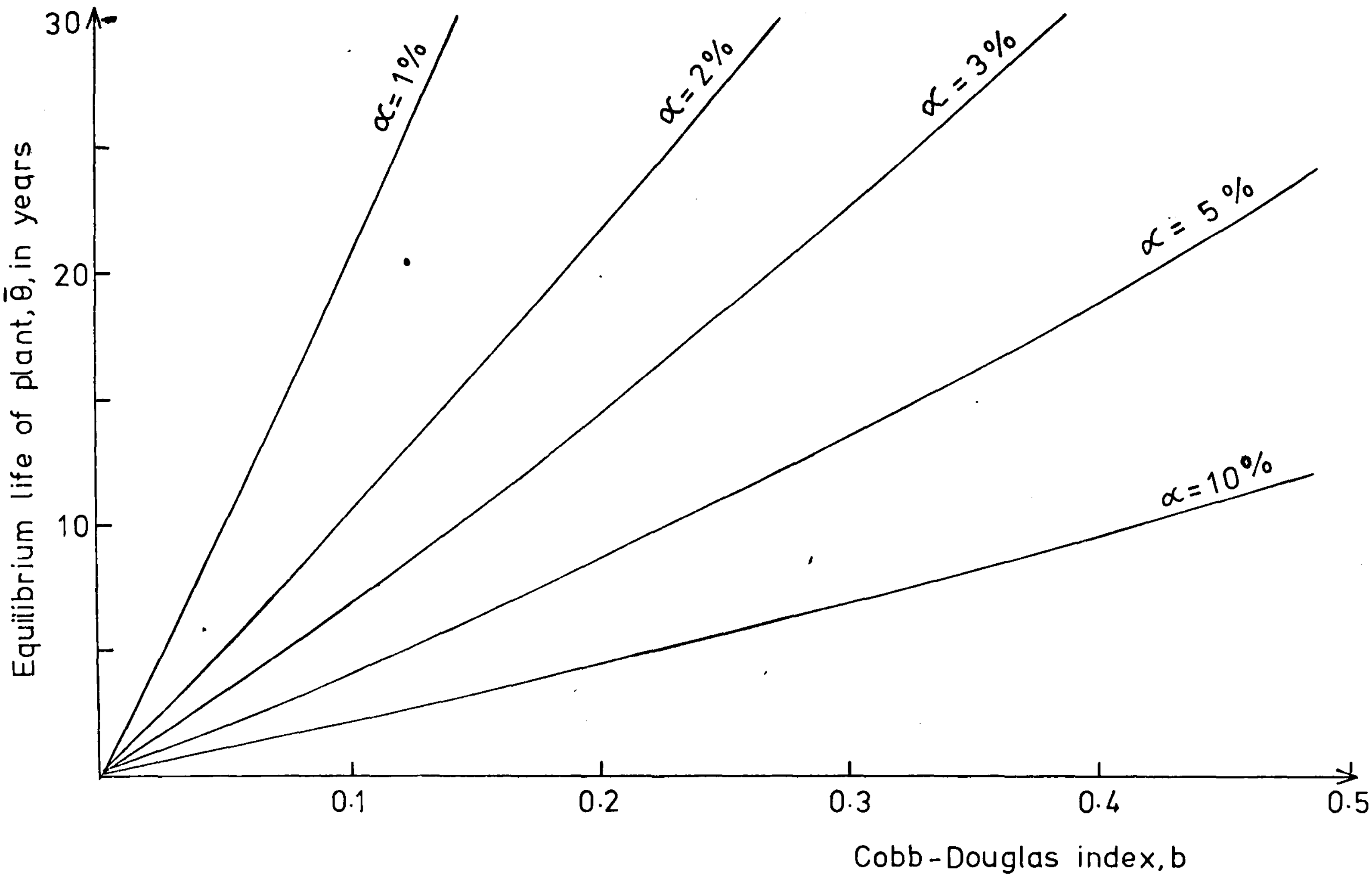


FIGURE 10 Equilibrium of Putty-Clay model (special case solution)

of the form $\bar{r} = \alpha + \varepsilon$ may be sought for some small ε, then the analysis of first order terms yields

$$\frac{\varepsilon}{\alpha} = 1 - \frac{bs_p}{b - (1 - b)s_w} = \frac{b(1 - s_p) - (1 - b)s_w}{b - (1 - b)s_w}.$$

We observe that both positive and negative ε may occur depending upon the savings rates, so that α is not a maximal value for $\bar{r}$ in this sense. Specifically if the savings rates are decreased from values which satisfy the condition of eqn. *(4.49)* then, assuming that $b - (1 - b)s_w > 0$, the rate of return $\bar{r}$ will be greater than α. This would seem to support a supply and demand theory for savings. With $s_p = 0{\cdot}55$, $s_w = 0{\cdot}08$, $b = 0{\cdot}25$ and $\alpha = 3\%$, $\bar{r}$ was computed as $7{\cdot}0\%$.

With the establishment of these models which generally are characterised by the possession of a unique balanced growth equilibrium solution we now turn to the questions of disequilibrium dynamics.

CHAPTER 5

THE STABILITY OF BALANCED GROWTH

The accepted concept of asymptotic stability of an equilibrium state $\underline{x}_e$ of the equation

$$\underline{\dot{x}} = F(t, \underline{x})$$

is that every solution starting at a state $\underline{x}_0$ sufficiently near to $\underline{x}_e$ converges to $\underline{x}_e$ as t increases indefinitely. For a system without delays the situation is simplified in that for a given $\underline{x}_0$ and a given t_0 the solution is unique; but for a *DDE*,

$$\underline{\dot{x}} = F(t, \underline{x}(*)), \tag{5.1}$$

this situation no longer pertains. In the latter case a solution of eqn. *(5.1)* denoted by $\underline{\psi}(t)$ can only be determined when $\underline{\psi}$ is initialised on some interval $[t_0 - q, t_0]$ where $q > 0$.

Accordingly we define the supremum (least upper bound) norm of $\underline{\psi}$ as

$$||\underline{\psi}(t)|| = sup_{s\epsilon[-q,0]} |\underline{\psi}(t + s)|, \tag{5.2}$$

and adopt the following definition for stability. $\underline{\psi}(t)$ is assumed to be a continuously differentiable function defined for all $t \in [t_0 - q, \infty)$. For $\delta > 0$, $\underline{x}_e$ is said to be exponentially stable if there exists a $D > 0$ and $d > 0$ such that

$$||\underline{\psi}(t_0) - \underline{x}_e|| < \delta$$

implies

$$|\underline{\psi}(t) - \underline{x}_e| \leq D \exp\{-d(t - t_0)\} \, ||\underline{\psi}(t_0) - \underline{x}_e||$$

for $t > t_0$.

This chapter is an investigation of the stability of the equilibrium states found to characterise the vintage and putty-clay models of the previous chapter. Each model which can be represented by eqn. *(5.1)* is rewritten in the form

$$\underline{\delta' x} = F_L(\delta\underline{x}(*)) + F_R(\underline{\delta x}(*)) \qquad (5.3)$$

where $\underline{x}(*)$ has been replaced by $\underline{x}_e + \underline{\delta x}(*)$, and $F_L(\underline{\delta x}(*))$ is the result of linearising $F(\underline{x}(*))$ about the equilibrium state $\underline{x}_e$ and setting $F(\underline{x}_e) = 0$ (as in fact none of the functions considered below is directly dependent upon t, the time variable is dropped from henceforth). F_L is then linear in $\underline{\delta x}(*)$ and $F_R(\underline{\delta x}(*))$ is the residual function $F(\underline{x}(*)) - F_L(\underline{\delta x}(*))$ which can be made arbitrarily small in the sup norm given by eqn. *(5.2)* on the interval $[t_0, \infty)$. That is to say we can find a $\delta > 0$ such that for all solutions $\underline{\delta x} = \underline{\psi}$ where $||\underline{\psi}|| < \delta$

$$|F_R(\underline{\psi})| \leq \varepsilon \, ||\underline{\psi}||$$

for all $t > t_0$ and arbitrarily small $\varepsilon > 0$.

The next step is to investigate the stability of the linearised system

$$\underline{\delta' x} = F_L(\delta\underline{x}(*)) \qquad (5.4)$$

using Nyquist and Pontryagin stability criteria for time delayed systems. The following question then arises. Can it be assumed, if stability at $\underline{x}_e$ can be proved under certain conditions for the linearised system of eqn. *(5.4)*, that the original system represented by eqn. *(5.1)* is also stable under the same conditions? Fortunately the result is affirmative as has been shown by Grossman[24].

5.1 Linearisation of the Vintage Model IIIA

The system equations for the vintage model with population growth are eqns. *(4.11)*, *(4.24)* and *(4.26)* which, reproduced in the form of eqn. *(5.1)*, are

$$\left.\begin{aligned}
\alpha\sigma \frac{dn}{dt} &= \alpha\left[s_p - (\alpha + \beta)\sigma + (s_w - s_p)\ exp\{-\alpha\theta\}\right]n \\
&\quad + exp\{-\alpha\theta\}\left[(s_w - s_p)\alpha v - s_w\ exp\{-\beta\theta\}n^*\right] \\
&\quad \times \left(- \gamma + \frac{\rho}{1 - v}\right), \\
\alpha \frac{dv}{dt} &= \alpha n - \alpha\beta v - exp\{-\beta\theta\}n^* \left(-\gamma + \frac{\rho}{1 - v}\right), \\
\alpha \frac{d\theta}{dt} &= (\alpha + \gamma) - \frac{\rho}{1 - v},
\end{aligned}\right\} \qquad (5.5)$$

where $n^* \equiv n(t - \theta)$ represents the term with lagged argument. Now $\underline{x} = [n, v, \theta]'$ is replaced by $\underline{x_e} + \underline{\delta x}$ where $\underline{x_e} = [\bar{n}, \bar{v}, \bar{\theta}]'$ and the linearised equations are then given by

$$\underline{\delta' x} = \frac{\partial F}{\partial x_1}\ \delta x_1 + \frac{\partial F}{\partial x_2}\ \delta x_2 + \frac{\partial F}{\partial x_3}\ \delta x_3$$

where $\underline{\delta x} = [\delta n, \delta v, \delta\theta]'$, and each of the partial derivatives is evaluated at $\underline{x} = \underline{x_e}$. This gives

$$\left.\begin{aligned}
\alpha\delta' n &= a_{11}\delta n - b_{11}\delta n^* + a_{12}\delta v + a_{13}\delta\theta \\
\alpha\delta' v &= \alpha\delta n - \alpha\ exp\{-\beta\bar{\theta}\}\delta n^* + a_{22}\delta v + \alpha\beta\ exp\{-\beta\bar{\theta}\}\bar{n}\delta\theta \\
\alpha\delta'\theta &= a_{32}\delta v
\end{aligned}\right\} \qquad (5.6)$$

where

$$a_{11} = \frac{\alpha}{\sigma}\,[s_p - (\alpha + \beta)\sigma + (s_w - s_p)\,exp\{-\alpha\bar{\theta}\}],$$

$$b_{11} = \frac{1}{\sigma}\,s_w\,exp\{-(\alpha + \beta)\bar{\theta}\}(-\gamma + \frac{\rho}{1 - \bar{v}})$$

$$= \frac{\alpha}{\sigma}\,s_w\,exp\{-(\alpha + \beta)\bar{\theta}\} > 0,$$

$$a_{12} = \frac{\alpha^2}{\sigma}\,(s_w - s_p)\,exp\{-\alpha\bar{\theta}\} + \frac{1}{\sigma}\frac{\rho}{1 - \bar{v}^2}\,exp\{-\alpha\bar{\theta}\}$$

$$\times\,[(s_w - s_p)\alpha\bar{v} - s_w\,exp\{-\beta\bar{\theta}\}\bar{n}],$$

$$a_{13} = -\frac{\alpha^2}{\sigma}\,(s_w - s_p)\,exp\{-\alpha\bar{\theta}\}\bar{n} - \frac{\alpha^3}{\sigma}\,(s_w - s_p)\bar{v}\,exp\{-\alpha\bar{\theta}\}$$

$$+ \frac{\alpha}{\sigma}\,s_w(\alpha + \beta)\,exp\{-(\alpha + \beta)\bar{\theta}\}\bar{n}$$

$$= -\frac{\alpha^2}{\sigma}\,(\bar{n} + \alpha\bar{v})(s_w - s_p)\,exp\{-\alpha\bar{\theta}\} + (\alpha + \beta)\bar{n}b_{11},$$

$$a_{22} = -\alpha\beta - \bar{n}\,\frac{\rho}{(1 - \bar{v})^2}\,exp\{-\beta\bar{\theta}\} \text{ and } a_{32} = \frac{-\rho}{(1 - \bar{v})^2} < 0.$$

Again $\delta n^* \equiv \delta n(t - \theta)$ has a lagged argument. In writing down the characteristic equation of the linear system of eqns. (5.6) $\delta n(t - \theta)$ is replaced by $\delta n(t)\,exp\{-s\theta\}$ or equivalently $\delta n\,exp\{-s\bar{\theta}\}$ to first order. Thus the characteristic equation is

$$det\begin{vmatrix} a_{11}-b_{11}exp\{-s\bar{\theta}\}-\alpha s, & a_{12}, & a_{13} \\ \alpha(1-exp\{-(s+\beta)\bar{\theta}\}), & a_{22}-\alpha s, & \alpha\beta exp\{-\beta\bar{\theta}\}\bar{n} \\ 0 & a_{32}, & -\alpha s \end{vmatrix} = 0. \tag{5.7}$$

Expanding the determinant one obtains an expression of the form

$$p(s) + exp\{-s\bar{\theta}\}q(s) = 0, \tag{5.8}$$

where $p(s)$ and $q(s)$ are polynomials in s of degrees 3 and 2 respectively. That is

$$p(s) = (a_{11} - \alpha s)\begin{vmatrix} (a_{22} - \alpha s) & \alpha\beta\, exp\{-\beta\bar{\theta}\}\bar{n} \\ a_{32} & -\alpha s \end{vmatrix} - \alpha \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & -\alpha s \end{vmatrix}$$

$$= (a_{11} - \alpha s)(\alpha^2 s^2 - \alpha a_{22} s - \alpha\beta\, exp\{-\beta\bar{\theta}\}\bar{n} a_{32})$$

$$- \alpha\ (- \alpha a_{12} s - a_{13} a_{32})$$

$$= - \alpha^3 s^3 + \alpha^2 (a_{11} + a_{22}) s^2 + \alpha[-a_{11} a_{22}$$

$$+ \alpha\beta\, exp\{-\beta\bar{\theta}\}\bar{n} a_{32} + \alpha a_{12}] s \qquad (5.9a)$$

$$+ \alpha[- \beta\, exp\{-\beta\bar{\theta}\}\bar{n} a_{11} + a_{13}] a_{32},$$

and

$$q(s) = -b_{11}(\alpha^2 s^2 - \alpha a_{22} s - \alpha\beta\, exp\{-\beta\bar{\theta}\}\bar{n} a_{32})$$

$$+ \alpha\, exp\{-\beta\bar{\theta}\}(- \alpha a_{12} s - a_{13} a_{32})$$

$$= - \alpha^2 b_{11} s^2 + \alpha(a_{22} b_{11} - \alpha\, exp\{-\beta\bar{\theta}\} a_{12}) s \qquad (5.9b)$$

$$+ \alpha\, exp\{-\beta\bar{\theta}\}(\beta\bar{n} b_{11} - a_{13}) a_{32}.$$

Denoting the coefficients of s^k in $p(s)$ and $q(s)$ by p_k and q_k respectively it follows that

$$p_0 + q_0 = \alpha\left[\beta\, exp\{-\beta\bar{\theta}\}\bar{n}(b_{11} - a_{11}) + a_{13}(1 - exp\{-\beta\bar{\theta}\})\right] a_{32}.$$

Now

$$b_{11} - a_{11} = \frac{\alpha}{\sigma}\left[s_w\, exp\{-(\alpha + \beta)\bar{\theta}\} - s_p + (\alpha + \beta)\sigma - (s_w - s_p)\, exp\{-\alpha\bar{\theta}\}\right]$$

which by the equilibrium equation for $\bar{\theta}$, eqn. *(4.28)*

$$= \frac{\alpha}{\sigma}\, exp\{-\alpha\bar{\theta}\}(s_w - s_p)\, \frac{\alpha}{\beta}\, (1 - exp\{-\beta\bar{\theta}\}),$$

hence on substitution for a_{11}, b_{11} and a_{13}

$$p_0 + q_0 = \frac{\alpha^2}{\sigma}\,\bar{n}(1 - \exp\{-\beta\bar{\theta}\})\Big[\alpha\,\exp\{-(\alpha+\beta)\bar{\theta}\}(s_w - s_p)$$

$$- \alpha\left(1 + \frac{\alpha}{\beta} - \frac{\alpha}{\beta}\exp\{-\beta\bar{\theta}\}\right)(s_w - s_p)\,\exp\{-\alpha\bar{\theta}\}$$

$$+ (\alpha+\beta)s_w\,\exp\{-(\alpha+\beta)\theta\}\Big]a_{32}$$

where eqn. *(4.27)* has been used to substitute for $\bar{v}$ in the expression for a_{13}. Further simplification gives

$$p_0 + q_0 = \frac{\alpha^2}{\sigma}\,\bar{n}(1 - \exp\{-\beta\bar{\theta}\})\,\exp\{-\alpha\bar{\theta}\}$$

$$\times \Big[(\alpha+\beta)s_w\,\exp\{-\beta\bar{\theta}\} - \alpha(s_w - s_p)\left(1 + \frac{\alpha}{\beta}\right)$$

$$\times (1 - \exp\{-\beta\bar{\theta}\})\Big]a_{32}.$$

Since $a_{32} < 0$ and assuming that $s_p \geq s_w$ we note that

$$p_0 + q_0 \begin{cases} < 0 & \text{if} \quad \beta > 0 \quad \text{or} \quad \beta < -\alpha < 0 \\ = 0 & \text{if} \quad \beta = 0 \quad \text{or} \quad \alpha + \beta = 0 \\ > 0 & \text{if} \quad -\alpha < \beta < 0. \end{cases} \tag{5.10}$$

Also

$$q_0 = \alpha\,\exp\{-\beta\bar{\theta}\}(\beta\bar{n}b_{11} - a_{13})a_{32}$$

$$= \alpha\,\exp\{-\beta\bar{\theta}\}\left[\frac{\alpha^2}{\sigma}\,(\bar{n} + \alpha\bar{v})(s_w - s_p)\,\exp\{-\alpha\bar{\theta}\} - \alpha\bar{n}b_{11}\right]a_{32}.$$

Again since $a_{32} < 0$ and $b_{11} > 0$ and assuming that $s_p \geq s_w$, then

$$q_0 > 0 \quad \text{for all } \beta.$$

It follows from eqn. *(5.10)* that

$$p_0 < 0 \quad \text{if} \quad \beta \geq 0 \quad \text{or} \quad \alpha + \beta \leq 0,$$

although the condition on p_0 in fact holds for all β. For if $-\alpha < \beta < 0$ then

$$a_{11} = \frac{\alpha}{\sigma}\left[s_p - (\alpha + \beta)\sigma + s_w - s_p\,exp\{-\alpha\bar{\theta}\}\right],$$

$$= -\frac{\alpha}{\sigma}\left[(s_w - s_p)\,\frac{\alpha}{\beta}\,(1 - exp\{-\beta\bar{\theta}\}) - s_w\,exp\{-\beta\bar{\theta}\}\right] exp\{-\alpha\bar{\theta}\}$$

by the equilibrium eqn. *(4.28)*

$$> 0 \quad \text{for} \quad \beta < 0 \quad \text{if} \quad s_p \geq s_w.$$

Hence

$$p_0 = \alpha\left[-\beta\,exp\{-\beta\bar{\theta}\}\bar{n}a_{11} + a_{13}\right]a_{32} < 0$$

for $-\alpha < \beta < 0$ since also $a_{13} > 0$ and $a_{32} < 0$ (see eqns. *(5.6)*). Hence we conclude that

$$q_0 > 0 \quad \text{and} \quad p_0 < 0 \quad \text{for all } \beta. \tag{5.11}$$

In completing general observations on the linearised system and its characteristic equation we also note that both leading coefficients p_3 and q_2 are negative in the form presented above.

5.2 Nyquist Analysis for the Vintage Model

The Nyquist stability criterion[49, 56] is derived from the encirclement theorem of complex variable theory. Let $B(s)$ be a function of the complex variable $s = \sigma + j\omega$ of the form

$$B(s) = e^{-\lambda s}\,\frac{(s - Z_1)(s - Z_2)\quad\ldots\quad(s - Z_m)}{(s - P_1)(s - P_2)\quad\ldots\quad(s - P_n)} \tag{5.12}$$

where $s = Z_1, Z_2, \ldots, Z_m$ are zeros and $s = P_1, P_2, \ldots, P_n$ are poles of $B(s)$ and λ is a constant. Now if C is a simple* closed

* *A contour is simple if it does not intersect itself.*

contour in the s-plane then C will be mapped by the function represented in eqn. *(5.12)* into a corresponding contour Γ in the $B(s)$-plane. If C is traversed once in the clockwise direction then the change in argument of the complex number represented by the image on the Γ contour is given by

$$\Delta arg\{B(s)\} = \Delta arg\{e^{-\lambda s}\} + \sum_{r=1}^{m} arg\{s - z_r\} - \sum_{r=1}^{n} arg\{s - p_r\}. \tag{5.13}$$

Now $arg\{e^{-\lambda s}\}$ is simply $-\lambda\omega$ and hence as C is closed

$$\Delta arg\{e^{-\lambda s}\} = 0.$$

Also $(s - z_r)$ can be interpreted geometrically as the vector from z_r to s in the s-plane. Thus $\Delta arg\{s - z_r\} = 0$ unless z_r lies within the contour C in which case $\Delta arg\{s - z_r\} = -2\pi$. Hence if the number of zeros of $B(s)$ which lie within C is Z (and assuming that none lie on C) then

$$\sum_{r=1}^{m} \Delta arg\{s - z_r\} = -2\pi Z.$$

By a similar argument

$$\sum_{r=1}^{n} \Delta arg\{s - p_r\} = -2\pi P,$$

where P is the number of poles of $B(s)$ within C. In consequence it may be deduced from eqn. *(5.13)* that

$$\Delta arg\{B(s)\} \;/\; 2\pi = P - Z. \tag{5.14}$$

The left-hand side of eqn. *(5.14)* may be interpreted as the number of counter-clockwise encirclements, N of the origin made by the Γ

coutour (as C is traversed once in the clockwise direction).

Now the linear system to be investigated, as represented by eqns. *(5.6)* will be stable if and only if the zeros of its characteristic equation *(5.8)* have negative real part. Eqn. *(5.8)* may be expressed in the form

$$1 + \exp\{-s\bar{\theta}\}\,\frac{q(s)}{p(s)} = 0, \tag{5.15}$$

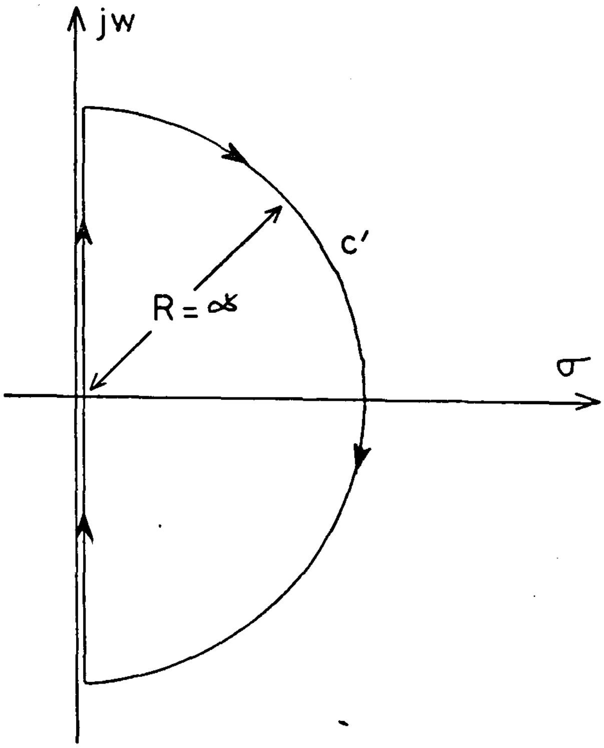
or in short

$$1 + G(s) = 0.$$

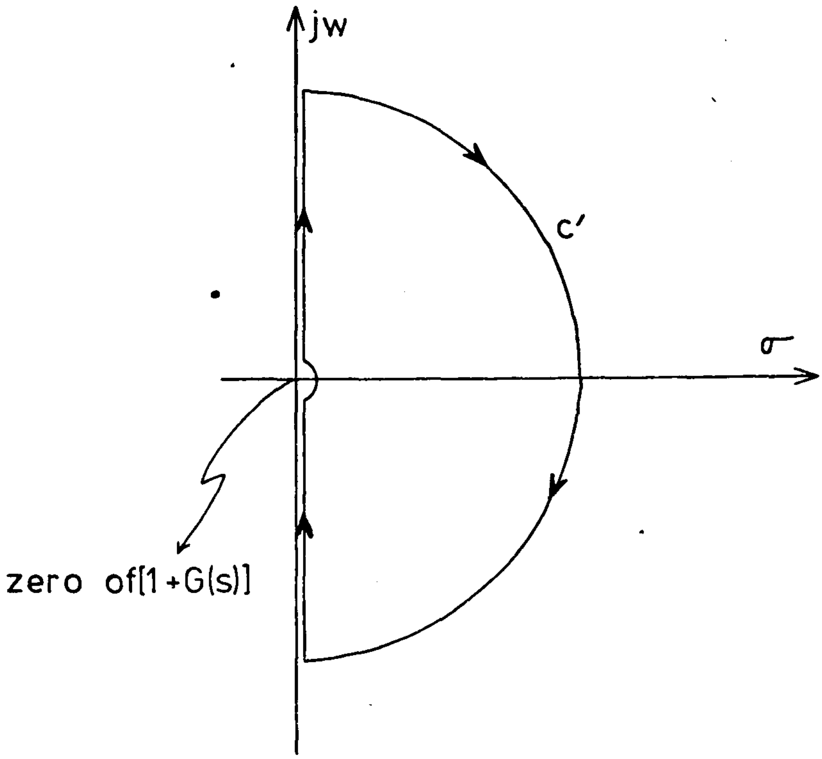
To investigate the number of zeros, if any, of $[1 + G(s)]$ which lie in the right hand half plane we choose as the contour C the special *stability* contour C' as shown in fig. 11 (a), and apply the result of eqn. *(5.14)*. The contour C' consists of the whole of the imaginary axis from $-\infty$ to $+\infty$, together with the infinite semi-circular arc which encompasses the entire right half of the s-plane. If any poles or zeros should lie on the imaginary axis then a small indentation is made to the contour at that point as shown in fig. 11 (b).

However, rather than substitute $B(s) = 1 + G(s)$ in the encirclement theorem, we note that $[1 + G(s)]$ and $G(s)$ have the same poles and that the origin in the $[1 + G(s)]$-plane corresponds to the point $(-1, 0)$ in the $G(s)$-plane. Hence by considering the map of C' in the $G(s)$-plane but by interpreting N as the number of encirclements of the point $(-1, 0)$ made by the Γ contour as C' is traversed once, the equation

$$N = P - Z$$



(a)



(b)

FIGURE 11 Nyquist stability contours.

gives the number of zeros of $[1 + G(s)]$ in the right half plane. P can be interpreted as the number of poles of $G(s)$ in the right half plane. $G(s)$ is known as the open loop transfer function in systems terminology. A necessary and sufficient condition for stability is $Z = 0$ $(N = P)$, and this is the Nyquist stability criterion which is now applied to the vintage model.

Where it is necessary to determine explicitly the map in the $G(s)$-plane of the Nyquist stability contour program SA3A is used. This program, which is listed and explained in Appendix 2, not only computes any number of points on the Nyquist map but also computes P, the number of poles in the right half plane, using the Routh-Hurwitz algorithm (for details see Ogata[48]). Some typical half maps for the vintage model with variable population growth are shown in fig. 12.

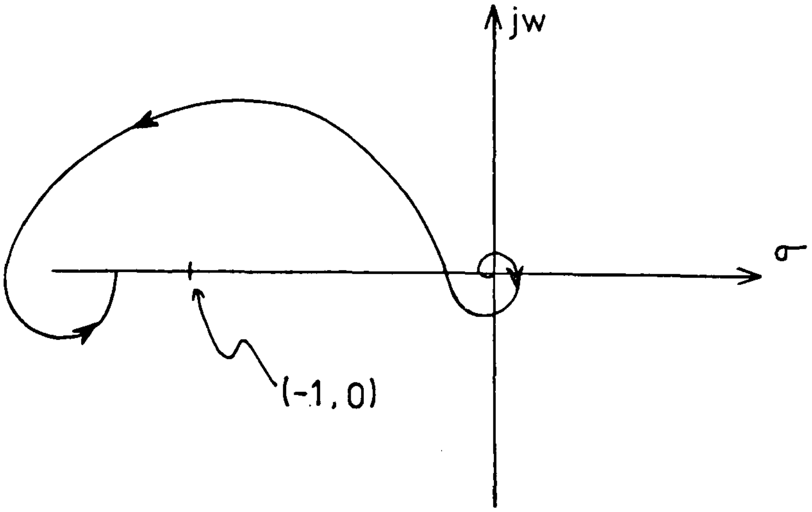
In the particular case of $\beta = 0$ when the characteristic equation has a zero at the origin and when we use the stability contour of fig. 11 (b), the map of the indentation can be determined by letting its equation be $s = \varepsilon\ exp\{j\psi\}$ where ε is small and ψ varies from $-\pi/2$ to $+\pi/2$. Then the map is given by

$$z = exp\{-\varepsilon\ exp(j\psi)\bar{\theta}\}\ \frac{q_0 + q_1\varepsilon\ exp\{j\psi\} + O(\varepsilon^2)}{p_0 + p_1\varepsilon\ exp\{j\psi\} + O(\varepsilon^2)}$$

$$= \left[1 - \varepsilon\ exp\{j\psi\}\bar{\theta}\right]\left[-1 + \frac{q_1 + p_1}{p_0}\ \varepsilon\ exp\{j\psi\}\right] + O(\varepsilon^2)$$

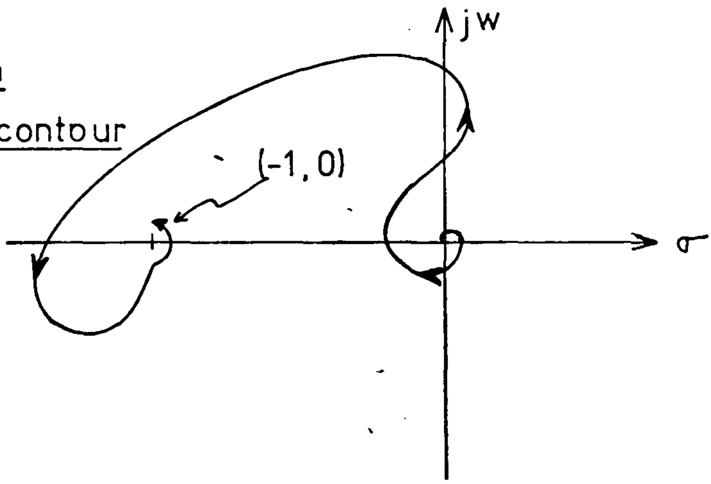
$$= -1 + \varepsilon(\bar{\theta} + \frac{q_1 + p_1}{p_0})\ exp\{j\psi\} + O(\varepsilon^2). \tag{5.16}$$

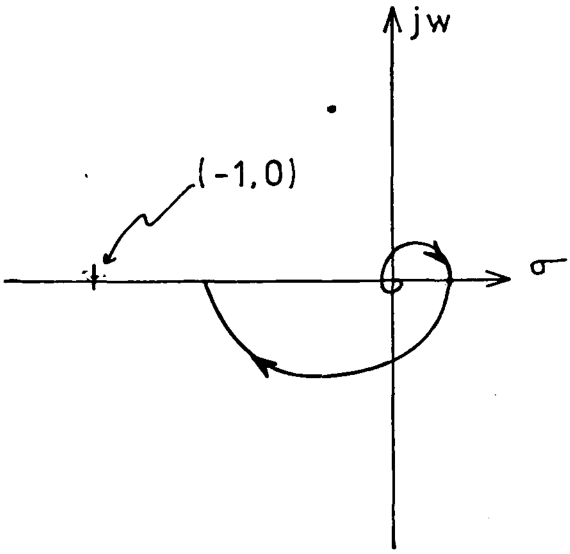
Now



(a) Case N=1

(b) Case N=2 (with indentation of contour at (-1, 0))





(c) Case N = 0

FIGURE 12 Typical Nyquist half-maps of the open loop transfer function for the linearised Vintage

$$(q_1 + p_1)/\alpha = a_{22}b_{11} - \alpha\ exp\{-\beta\bar{\theta}\}a_{12} - a_{11}a_{22}$$
$$+ \alpha\beta\ exp\{-\beta\bar{\theta}\}\bar{n}a_{32} + \alpha a_{12},$$
$$= a_{22}(b_{11} - a_{11}) \quad \text{when} \quad \beta = 0,$$
$$= -\frac{\alpha}{\sigma}\,[s_p(1 - exp\{-\alpha\bar{\theta}\}) - \alpha\sigma]a_{22},$$
$$= -\frac{\alpha}{\sigma}\,(s_p - s_w)\alpha\bar{\theta}\ exp\{-\alpha\bar{\theta}\}a_{22}$$

by the equilibrium eqn. *(4.21)*. Hence by substitution

$$\bar{\theta} + \frac{q_1 + p_1}{p_0} = \bar{\theta}\left[1 - \frac{\alpha^3(s_p - s_w)\ exp\{-\alpha\bar{\theta}\}a_{22}}{\sigma\alpha a_{13}a_{32}}\right],$$
$$= \bar{\theta}\left[1 - \frac{(s_p - s_w)\ exp\{-\alpha\bar{\theta}\}\bar{n}}{(\bar{n} + \alpha\bar{v})(s_p - s_w)\ exp\{-\alpha\bar{\theta}\} + \bar{n}s_w\ exp\{-\alpha\bar{\theta}\}}\right],$$
$$= \bar{\theta}\,\frac{\alpha\bar{\theta}(s_p - s_w) + s_w}{(1 + \alpha\bar{\theta})(s_p - s_w) + s_w} > 0 \quad \text{if} \quad s_p > s_w.$$

It then follows from eqn. *(5.16)*, since the coefficient of $\varepsilon\ exp\{j\psi\}$ is positive, that there is an indentation in the contour Γ at the point *(-1, 0)* in the *G(s)*-plane which is a semi-circular arc traversed counter-clockwise from the lower to the upper half plane. This situation is as depicted in fig. 12 (b).

The results of the Nyquist analysis are now evaluated for three separate cases.

5.2.1 Case $\beta < 0$ $(\alpha + \beta > 0)$

The Nyquist map for this first case is as in fig. 12 (a). In particular when $s = 0$ the open loop transfer function

$$G(0) = \frac{q_0}{p_0} < -1$$

since $p_0 + q_0 > 0$ and $p_0 < 0$ from eqns. *(5.10)* and *(5.11)*. Also as $s \to \infty$ along the imaginary axis

$$G(s) \to exp\{-j\omega\bar{\theta}\}\frac{q_2}{p_3 s},$$

which represents a clockwise spiral to the origin in the $G(s)$-plane. Thus, because of the symmetry, we may conclude that the number of encirclements of $(-1, 0)$ is ODD (typically $N = 1$). Furthermore since $p_0 < 0$ and $p_3 < 0$ and $p(s)$ is cubic, it is evident that $P = 0$ or *2*; that is P is EVEN. Application of the Nyquist criteria tells us that Z is non-zero so that equilibrium is UNSTABLE.

What then are the implications of this result? Certainly it is evident that active control policies will be essential to maintain balanced growth, although this is by no means the full picture as the result applies purely to the local neighbourhood of the equilibrium. Investigation of the global situation in the next chapter produces further results of relevance to the control problem. It is interesting to note in passing that the U.K. economy has experienced a marginal decline in its work force over some recent years (see Central Statistical Office[12]).

5.2.2 Case $\beta = 0$

Unfortunately the present case cannot be treated so neatly as that above and numerical computation of the Nyquist path is necessary requiring the assumption of specific parameter values. The computation has been undertaken with various sets of such parameter values and the diagnosis has always been STABLE. The choice of parameter values is discussed in the next chapter, but the following set has frequently been used and will serve to illustrate the result:

$$\left.\begin{aligned}
s_W &= 0{\cdot}08, \\
s_P &= 0{\cdot}5, \\
\sigma &= 3{\cdot}6, \\
\alpha &= 0{\cdot}03 \text{ (3\% growth per annum)}, \\
\gamma &= 0{\cdot}02, \\
\rho &= 0{\cdot}001.
\end{aligned}\right\} \qquad (5.17)$$

The map in this case is as shown in fig. 12 (b), which was obtained from the contour shown in fig. 11 (b) since the system characteristic equation has a zero root ($p_0 + q_0 = 0$). The map of the indentation has already been discussed above and the asymptotic behaviour as $s \to \infty$ along the Nyquist contour is as for $\beta < 0$. Fig. 12 (b) shows that $N = 2$ and as $P \geq N$ (since Z must be non-negative) we must have $P = 2$ also, from which the result follows. Again the result must be qualified as being a purely local phenomena although potentially more favourable than for $\beta < 0$.

5.2.3 <u>Case $\beta > 0$</u>

For $\beta > 0$ the Nyquist path as computed with the parameter values of eqns. *(5.17)* and various β in the range $0 < \beta \leq 5\%$ is generally of the type illustrated in fig. 12 (c), where the point $(-1, 0)$ is never encircled. In this case it is also necessary to compute the value of P using the Routh-Hurwitz method. The outcome is again STABLE.

The Nyquist criterion is thus seen to be a useful tool for the economic analyst, which can be applied to any such linearised system as was obtained from the vintage model. In the case of the putty-clay model however an alternative technique is demonstrated which uses a theorem due to Pontryagin applicable in special circumstances which pertain in this model.

5.3 Linearisation of the Putty-Clay Model IV

The following analysis is undertaken for the classical savings case only where $s_w = 0$ and $s_p = 1$. Under these circumstances the equilibrium solution found in Sec. 4.7.1 applies in which $\bar{r} = \alpha$, $\bar{\phi} = \bar{\theta}$ and

$$exp\{\alpha\bar{\theta}\} = 1 + \frac{\alpha\bar{\theta}}{1 - b} \qquad (4.50)$$

For this special case a further minor simplification of the system equations is undertaken prior to linearisation.

From eqns. *(4.43)* we derive the two equations

$$(1 - b)\sigma c^{1-b} = \frac{b}{r - \lambda}\,(exp\{-\lambda\phi\} - exp\{-r\phi\}) \qquad (5.18a)$$

and

$$\sigma c^{1-b} = \frac{b}{r}\,(1 - exp\{-r\phi\}). \qquad (5.18b)$$

In the first of eqns. *(4.42)* we set the classical savings conditions and differentiate out,

$$\sigma\,\frac{d}{dt}\,(cn\;exp\{\alpha t\}) = \frac{d}{dt}\int_{t-\theta}^{t} c^b(\tau)n(\tau)\;exp\{\alpha\tau\}d\tau$$

$$-\;[bc^{-1}(t - \theta)c'(t - \theta) + \alpha]c^b(t - \theta)\;exp\{\alpha(t - \theta)\}(1 - \frac{d\theta}{dt})v$$

$$- c^b(t - \theta)\ exp\{\alpha(t - \theta)\}\ \frac{d}{dt}\int_{t-\theta}^{t} n(\tau)d\tau,$$

that is, now using the second of eqns. *(4.42)* and eqn. *(4.45)*

$$\sigma\left(\frac{dc}{dt}\,n + c\,\frac{dn}{dt} + \alpha cn\right) = c^b n - c^b(t - \theta)\ exp\{-\alpha\theta\}(n + \lambda v).$$

But

$$w(t) = \frac{Z(t - \theta)}{N(t - \theta)} = \frac{1}{\sigma}\,k^b(t - \theta)\ exp\{\alpha(1 - b)(t - \theta)\}$$

by the production function, eqn. *(4.30)*; that is

$$w(t) = \frac{1}{\sigma}\,c^b(t - \theta)\ exp\{\alpha(t - \theta)\} \tag{5.18c}$$

Hence from above

$$\sigma\left(\frac{dc}{dt}\,n + c\,\frac{dn}{dt} + \alpha cn\right) = c^b n - \sigma w\ exp\{-\alpha t\}(n + \lambda v). \tag{5.18d}$$

Eqn. *(4.45)* is reproduced in the form

$$\lambda = -\gamma + \frac{\rho}{1 - v} \tag{5.18e}$$

and $\lambda = \frac{1}{w}\frac{dw}{dt}.$ *(5.18f)*

Next we use the logarithmic form of eqn. *(4.44)* which gives

$$\lambda\phi = \alpha\theta + b(log\{c\} - log\{c(t - \theta)\}). \tag{5.18g}$$

Finally eqn. *(4.16)* is reproduced, that is

$$\frac{dv}{dt} = n - n(t - \theta)(1 - \frac{d\theta}{dt}). \tag{5.18h}$$

Eqns. *(5.18)* now represents the complete system in the vector

$\underline{x} = [n, v, \theta, c, \lambda, r, w\ exp\{-\alpha t\}, \phi]'$ which is linearised as was the vintage model in sec. 5.1 about the equilibrium

$$\underline{x_e} = [\bar{n}, \bar{n}\bar{\theta}, \bar{\theta}, \bar{c}, \alpha, \alpha, \bar{w}_0, \bar{\theta}]'.$$

The characteristic equation of the linearised system is given in determinant form in fig. 13. This is subsequently reduced to a 4×4 determinant equation by the elementary row operations:

$R1 \div b$, $R4 \div \sigma$, $R1 - \frac{1}{2}\bar{\theta}^2\ exp\{-\alpha\bar{\theta}\}R6$,
$R4 - \bar{w}_0\bar{n}\bar{\theta}R6$, $R5 - R6$ and $R7 - \bar{\theta}R6$ to
eliminate row 6 with column 5;

$R7 \times \bar{w}_0 + R3$, $R5 \div \rho/(1 - \bar{n}\bar{\theta})^2$, $R4 + \alpha\bar{w}_0R5$
and $R8 + sR5$ to eliminate row 5 with column 2;

$R7 \div \alpha\bar{w}_0$, $R1 + (1 - \alpha\bar{\theta})\ exp\{-\alpha\bar{\theta}\}R7$ and
$R2 \div b + exp\{-\alpha\bar{\theta}\}R7$ to eliminate row 7
with column 8; and lastly
$R1 \times \frac{1}{\alpha}\sigma\bar{c}^{1-b} - \frac{1}{2}\bar{\theta}^2\ exp\{-\alpha\bar{\theta}\}R2$ to eliminate
row 2 with column 6.

The reduced characteristic equation is of the form

$$\begin{vmatrix} 0 & 0 & a_{13} & a_{14}+b_{14}s \\ cs-\alpha\bar{w}_0\bar{\theta} & 0 & \bar{n}s+b_{23} & a_{24}+b_{24}s \\ -(1-exp\{-s\bar{\theta}\}) & -\bar{n}s & 0 & c_{34}s^2 \\ 0 & \alpha\bar{w}_0 & -\frac{b}{c}\bar{w}_0exp\{-s\bar{\theta}\} & 1 \end{vmatrix} = 0, \qquad (5.19)$$

where

ions	n	v	θ	c	λ	r	$w\exp\{-\alpha t\}$	φ
1	0	0	0	$(1-b)^2\sigma\bar{c}^{-b}$	$\frac{b}{2}\bar{\theta}^2\exp\{-\alpha\theta\}$	$\frac{b}{2}\bar{\theta}^2\exp\{\alpha\theta\}$	0	$-b(1-\alpha\bar{\theta}) \times \exp\{-\alpha\bar{\theta}\}$
2	0	0	0	$(1-b)\sigma\bar{c}^{-b}$	0	$\frac{b}{\alpha}\sigma\bar{c}^{1-b}$	0	$-b\exp\{-\alpha\theta\}$
3	0	0	$\alpha\dot{\bar{w}}_0$	$-\frac{b}{c}\bar{w}_0\exp\{-s\bar{\theta}\}$	0	0	1	0
4	$\sigma(\bar{c}s-\alpha\bar{w}_0\bar{\theta})$	$\sigma\bar{w}_0\alpha$	0	$(\sigma s+\sigma\alpha-b\bar{c}^{b-1})\bar{n}$	$\sigma\bar{w}_0\bar{n}\bar{\theta}$	0	$\sigma\bar{n}(1+\alpha\theta)$	0
det 5	0	$-\rho/(1-\bar{n}\bar{\theta})^2$	0	0	1	0	0	0
6	0	0	0	0	1	0	$-s/\bar{w}_0$	0
7	0	0	$-\alpha$	$-\frac{b}{\bar{c}}(1-\exp\{-s\bar{\theta}\})$	$\bar{\theta}$	0	0	α
8	$-(1-\exp\{-s\bar{\theta}\})$	s	$-\bar{n}s$	0	0	0	0	0

$= 0$

FIGURE 13 Characteristic equation for linearised Putty-Clay model (special case only)

$$a_{13} = \frac{1}{\alpha}\,\sigma\bar{c}^{1-b}\left[\frac{1}{b}\,(1-b)^2\sigma\bar{c}^{-b} - \frac{b}{\alpha\bar{c}}\,(1-\alpha\bar{\theta})\,exp\{-\alpha\bar{\theta}\}\right]$$

$$- \frac{1}{2}\,\bar{\theta}^2 exp\,\{-\alpha\bar{\theta}\}\left[\frac{1}{b}\,(1-b)\sigma\bar{c}^{-b} - \frac{b}{\alpha\bar{c}}\,exp\{-\alpha\bar{\theta}\}\right],$$

$$a_{14} = \frac{1}{\alpha}\,\sigma\bar{c}^{1-b}\left[\frac{1}{\alpha\bar{w}_0}\,(1-\alpha\bar{\theta})\,exp\{-\alpha\bar{\theta}\}\right]$$

$$- \frac{1}{2}\,\bar{\theta}^2\,exp\{-\alpha\bar{\theta}\}\left[\frac{1}{\alpha\bar{w}_0}\,exp\{-\alpha\bar{\theta}\}\right]$$

$$= \frac{1}{\alpha\bar{w}_0}\left[\frac{1}{\alpha}\sigma\bar{c}^{1-b}(1-\alpha\bar{\theta}) - \frac{1}{2}\,\bar{\theta}^2\,exp\{-\alpha\bar{\theta}\}\right]\,exp\{-\alpha\bar{\theta}\},$$

$$b_{14} = \frac{1}{\alpha}\,\sigma\bar{c}^{1-b}\,\frac{1}{2\bar{w}_0}\,\bar{\theta}^2\,exp\{-\alpha\bar{\theta}\} + \bar{\theta}a_{14},$$

$$b_{23} = (\alpha - \frac{1}{\sigma}\,b\bar{c}^{b-1})\bar{n},$$

$$a_{24} = \bar{n}(1 + \alpha\bar{\theta}),$$

$$b_{24} = n\bar{\theta} + \alpha(1 - \bar{n}\bar{\theta})^2/\rho,$$

and $$c_{34} = \frac{1}{\bar{w}_0}\,(1 - \bar{n}\bar{\theta})^2/\rho.$$

The coefficients in the first row are somewhat simplified using the identity

$$\sigma\bar{c}^{(1-b)} = \frac{b}{1-b}\,\bar{\theta}\,exp\{-\alpha\bar{\theta}\} \tag{5.20}$$

which is the steady state version of eqn. *(5.18a)*. We have

$$\frac{a_{13}}{\bar{\theta}\,exp\{-2\alpha\bar{\theta}\}} = \frac{1}{\alpha}\,\frac{b}{1-b}\left[\frac{(1-b)}{\bar{c}}\bar{\theta} - \frac{b}{\alpha\bar{c}}\,(1-\alpha\bar{\theta})\right]$$

$$- \frac{1}{2}\,\bar{\theta}\left[\frac{1}{\bar{c}}\,\bar{\theta} - \frac{b}{\alpha\bar{c}}\right],$$

$$= \left[\frac{1}{\alpha}\,\frac{b}{1-b} - \frac{1}{2}\,\bar{\theta}\right]\left[\frac{\bar{\theta}}{\bar{c}} - \frac{b}{\alpha\bar{c}}\right], \tag{5.21}$$

and similarly

$$\frac{a_{14}}{\bar{\theta}\,exp\{-2\alpha\bar{\theta}\}} = \frac{1}{\alpha\bar{w}_0}\left[\frac{1}{\alpha}\,\frac{b}{1-b}\,(1-\alpha\bar{\theta}) - \frac{1}{2}\,\bar{\theta}\right]$$

$$= \frac{1}{\alpha\bar{w}_0}\left[\frac{1}{\alpha} - \frac{1+b}{2b}\,\bar{\theta}\right]\frac{b}{1-b}$$

$$\frac{b_{14}}{\bar{\theta}\ exp\{-2\alpha\bar{\theta}\}} = \frac{\bar{\theta}}{\alpha\bar{w}_0}\left[\frac{\theta}{2} + \frac{1}{\alpha} - \frac{1+b}{2b}\bar{\theta}\right]\frac{b}{1-b}$$

$$= \frac{\bar{\theta}}{\alpha\bar{w}_0}\left[\frac{1}{\alpha} - \frac{\bar{\theta}}{2b}\right]\frac{b}{1-b} \qquad (5.22)$$

The expansion of eqn. *(5.19)* gives the characteristic equation in the form

$$p(s) + exp\{-s\bar{\theta}\}q(s) = 0 \qquad (5.23)$$

where on this occasion both $p(s)$ and $q(s)$ are cubic polynomials.

5.4 Stability of the Putty-Clay Model using Pontryagin's Criteria

The Pontryagin criteria is a non-graphical means of investigating the roots of an exponential characteristic polynomial such as eqn. *(5.23)*. For the purpose of applying the criteria, full details of which are to be found in Porter[56], eqn. *(5.23)* is rewritten in the form

$$F(s) = \hat{p}(s)\ exp\{s\bar{\theta}\} + \mu\hat{q}(s) = 0, \qquad (5.23a)$$

where the factor μ represents the ratio of the leading coefficient of $q(s)$ to that of $p(s)$ so that p cap and q cap each have a leading coefficient of unity.

Now if all the zeros of $F(s)$ lie to the left of the imaginary axis in the s-plane, the vectors joining the zeros to a point $s = j\omega$ on the imaginary axis will always rotate with positive (i.e. anti-clockwise) velocities as ω increases from $-\infty$ to $+\infty$. The latter is therefore a necessary condition for stability (Pontryagin also established sufficient conditions but these are not required in the present application). If $s = z_k$ is a zero of the characteristic equation then $F(z_k) = 0$ and let

$$F(j\omega) = P(\omega) + jQ(\omega). \qquad (5.24)$$

It follows, if the necessary condition for stability is to be satisfied, that the vector joining the origin to the point (P, Q) in the $F(s)$-plane must also rotate with positive angular velocity. Now

$$arg\{F(j\omega)\} = tan^{-1}\left\{\frac{Q(\omega)}{P(\omega)}\right\} \qquad (5.25)$$

and $\frac{d}{d\omega}\left[arg\{F(j\omega)\}\right]$ will always be positive if all the zeros of $F(s)$ have negative real part. Differentiation of eqn. (5.25) yields

$$\frac{d}{d\omega}\left[arg\{F(j\omega)\}\right] = \frac{P(\omega)Q'(\omega) - P'(\omega)Q(\omega)}{P^2(\omega) + Q^2(\omega)}$$

where the primes denote differentiation with respect to ω, and a necessary condition for stability is therefore that

$$P(\omega)Q'(\omega) - P'(\omega)Q(\omega) > 0 \qquad (5.26)$$

for all real values of ω.

As a corollary to the Pontryagin condition we derive below a result which is applicable in the circumstances of the putty-clay model.

Lemma: If $F(s)$ is an exponential characteristic polynomial of the form

$$p(s)\ exp\{\lambda s\} + \mu q(s) = 0$$

where $p(s)$ and $q(s)$ are polynomials of the same degree, k in s with leading coefficients of unity, and $\lambda > 0$ then the system is necessarily unstable if $|\mu| > 1$.

Proof: Let $s = j\omega$ and let $\omega \to \infty$ so that only leading coefficients need be considered. $F(s)$ is approximated by

$$F(s) = s^k(\exp\{\lambda s\} + \mu).$$

If k is even then

$$F(j\omega) = B\omega^k(\cos\{\lambda\omega\} + j\sin\{\lambda\omega\} + \mu)$$

where B takes the value $+1$ or -1. Now, if $F(j\omega) = P(\omega) + jQ(\omega)$ then from above

$$\left.\begin{aligned} P(\omega) &= B\omega^k(\cos\{\lambda\omega\} + \mu) \\ Q(\omega) &= B\omega^k \sin\{\lambda\omega\} \end{aligned}\right\}$$

and differentiation with respect to ω gives

$$\left.\begin{aligned} P'(\omega) &= B\omega^k\left(\frac{k}{\omega}\cos\{\lambda\omega\} + \frac{k}{\omega}\mu - \lambda\sin\{\lambda\omega\}\right) \\ Q'(\omega) &= B\omega^k\left(\frac{k}{\omega}\sin\{\lambda\omega\} + \lambda\cos\{\lambda\omega\}\right) \end{aligned}\right\}$$

which further approximates for large ω to

$$\left.\begin{aligned} P'(\omega) &= -B\omega^k\lambda\sin\{\lambda\omega\} \\ Q'(\omega) &= B\omega^k\lambda\cos\{\lambda\omega\} \end{aligned}\right\}$$

Hence $P(\omega)\,Q'(\omega) - P'(\omega)Q(\omega)$

$$= B^2\lambda\omega^{2k}\left[\cos\{\lambda\omega\}(\cos\{\lambda\omega\} + \mu) + \sin^2\{\lambda\omega\}\right]$$

$$= \lambda\omega^{2k}(1 + \mu\cos\{\lambda\omega\})$$

which takes negative values as $\omega \to \infty$ if $|\mu| > 1$. Hence the stability criterion of eqn. *(5.26)* is violated and the system is unstable.

If k is odd the corresponding functions are

$$F(j\omega) = Bj\omega^k(\cos\{\lambda\omega\} + j\sin\{\lambda\omega\} + \mu),$$

$$\left.\begin{aligned} P &= -B\omega^k \sin\{\lambda\omega\} \\ Q &= B\omega^k(\cos\{\lambda\omega\} + \mu), \end{aligned}\right\}$$

$$\left.\begin{aligned} P' &= -B\omega^k\lambda \cos\{\lambda\omega\} \\ Q' &= -B\omega^k\lambda \sin\{\lambda\omega\}, \end{aligned}\right\}$$

whence $P(\omega)Q'(\omega) - P'(\omega)Q(\omega)$

$$= B^2\lambda\omega^{2k}[\sin^2\{\lambda\omega\} + \cos\{\lambda\omega\}(\cos\{\lambda\omega\} + \mu)]$$

$$= \lambda\omega^{2k}(1 + \mu\cos\{\lambda\omega\})$$ as before.

The required result is thus *proved.*

In the light of this result we can now examine the stability of the putty-clay model as represented by the characteristic equation *(5.19)*. The leading coefficients are given by

$$p_3 = -a_{13}\bar{c}c_{34}\alpha\bar{w}_0$$

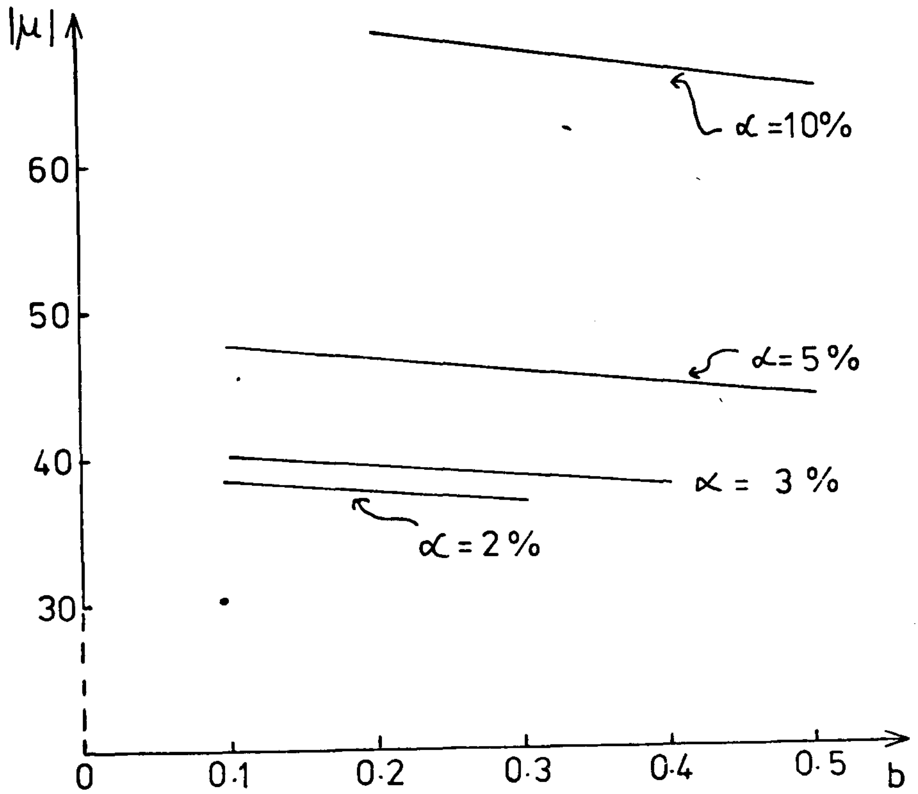
and $$q_3 = -b_{14}b\bar{\bar{n}}\bar{w}_0,$$

so that $$\mu = \frac{b}{\alpha}\frac{\bar{n}}{c}\frac{b_{14}}{a_{13}c_{34}}. \qquad (5.27)$$

Fig. 14 shows variations in the parameter μ for different values of α and b in the classical savings case. In all events the criterion $|\mu| > 1$ applies and as a result of the above lemma the system is UNSTABLE.

FIGURE 14 Variation in the stability parameter, μ for the Putty Clay model (special case)

($\mu \equiv$ ratio of leading coefficients in exponential characteristic equation)



5.5 Postscript

The foregoing chapter has demonstrated the application of a number of related techniques to examine the stability of balanced growth, and the results are summarised as follows:

$$\text{Vintage model} \begin{cases} \beta < 0 & \text{unstable} \\ \beta \geq 0 & \text{stable,} \end{cases}$$

$$\text{Putty-clay model} \qquad \text{unstable.}$$

Such results are clearly important in understanding the economic mechanism and providing effective control policy decisions. In the models considered the equilibrium solutions were unique but this says nothing about the possible existence of limit cycle solutions which may themselves be stable or unstable. It has not been possible to find any such limit cycle solutions for the vintage or putty-clay models, but the situation gives rise to further questions. If balanced growth is stable the economist must be concerned with the so called *domain of attraction* of the stable equilibrium and the nature of the response when the system is disturbed within this domain. Alternatively if balanced growth is unstable, then what sort of divergent response follows a disturbance. To provide the full answer to these questions for a complex non-linear system is virtually impossible, although some progress is achieved by employing the tool of simulation.

CHAPTER 6

DIGITAL SIMULATION

The advent of the modern digital computer with a cycle time per operation of a few micro-seconds or less has elevated simulation to a most important tool of analysis. It has already been indicated in the introductory chapter how this has led on the one hand to the development of large scale econometric models, although the present application is of a somewhat different character. We are able using simulation to study the effects of the complex non-linearities and time delays in the differential-delay models specified in chapter 4 without recourse to an analysis of limited cases or approximated models as for instance in the previous chapter.

Of course there are drawbacks. Firstly we are dealing with a real continuous system* which we have chosen to represent by a continuous model, albeit primarily for reasons of mathematical convenience. The technique of digital simulation requires that from some known initial state one advances in discrete time steps computing the new state of the model from the system equations at each stage. (By comparison analogüe computer simulation is continuous although the difficulty of representing non-linearities and time delays render it inappropriate for present purposes. Analogue computation was however used to produce the phase diagram for model II in fig. 6.) Thus in digital simulation of a continuous system the

* *Even this statement is contraversial for although agreggated variables such as output and wages behave as continuous variables they do of course represent the resultant of a great number of individual components some of which may vary in a discrete fashion.*

differential equations must be approximated by a difference equation equivalent, which naturally leads to questions regarding the accuracy of the solution obtained as a representation of the solution to the original continuous model. Intuition suggests that this problem can be resolved and any required degree of accuracy achieved by an appropriately small choice of time increment; but this is to ignore error build-up characteristics which can occur from one step to the next. If such characteristics do occur then the numerical algorithm used to approximate the model differential equations is said to be unstable. Hence for long run simulations it is necessary to choose an algorithm which is numerically stable as well as choosing an appropriate time increment or step length (of course too pessimistic a choice of step length will result in needless computation and expense).

A further drawback of simulation is the need to adopt specific parameter values at the outset. Not only does this pose the problem of estimation to which brief mention has already been made but also any results are conditioned by the particular parameter set to which they relate. In theory it is possible to vary the parameter set without limit but in practice this is obviously an area of compromise.

The present chapter explains how these problems have been tackled in implementing simulations of the vintage and putty-clay models on the University of Warwick 4130 (I.C.L.) computer. At the same time the methods used are generally relevant to the simulation of any system of differential-delay equations and equally any digital computer. Each program is documented and listed in appendix 2 in such a way that the results may be reproduced or extended by the reader. Additionally the programs are written in such a way as they may be easily amended for the simulation of alternative systems, economic and otherwise.

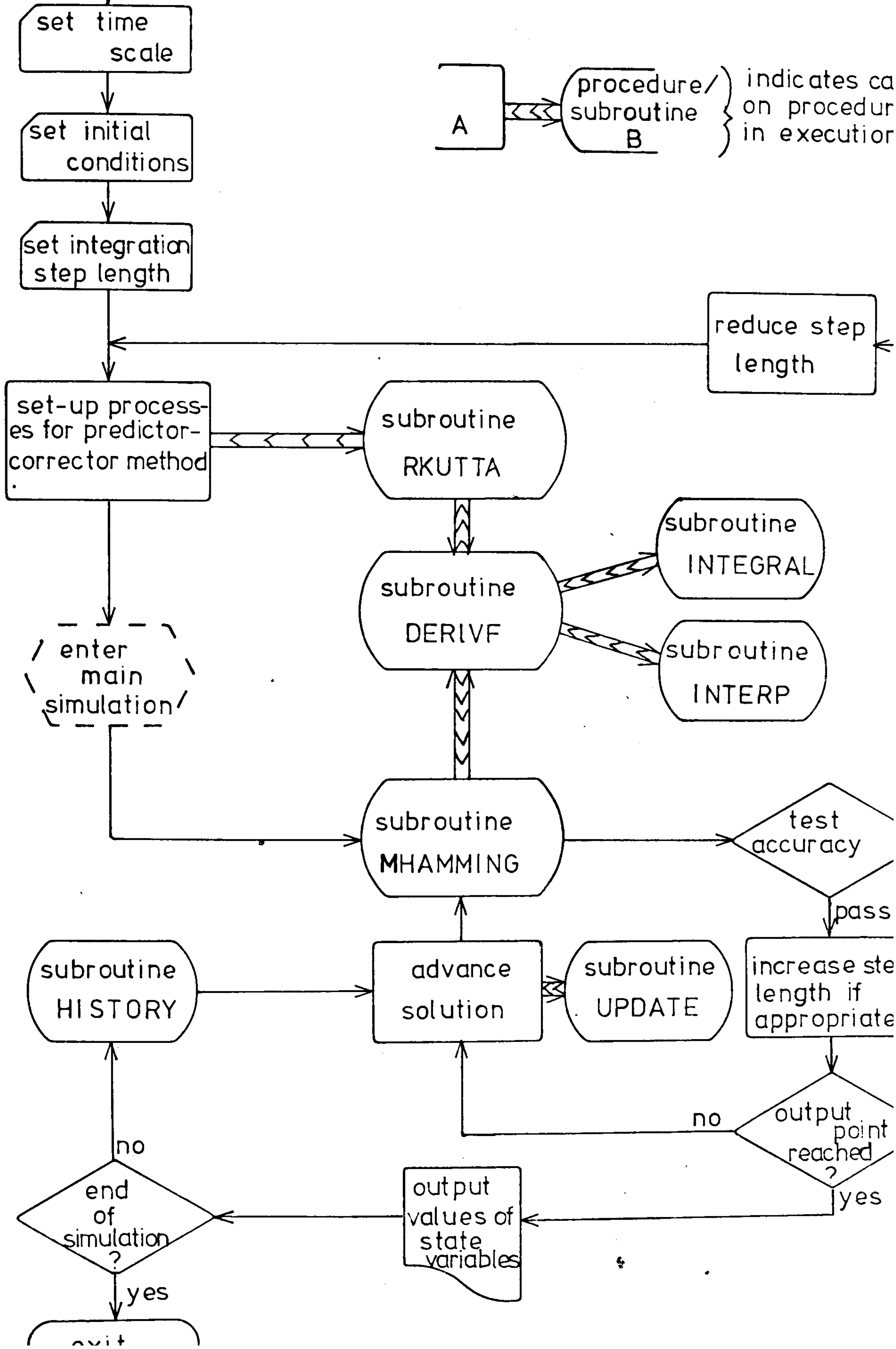
6.1 Numerical Algorithms for the Solution of a Differential-Delay System

The complexity of the systems to be simulated governs the need for efficient programs which will achieve the required solution with maximum speed. At this stage we are not concerned with the differences between the vintage and putty-clay models but it should be noted that in the latter case each evaluation of the system equations will involve the solution of the set of non-linear algebraic eqns. *(4.43)* and that this may be necessary several hundred times per simulated year. For these reasons a predictor-corrector integration process is used with a self-adaptive step length mechanism. Automatic step length selection is particularly necessary with the delay features that are present because disturbances which may result in rapid movement of the variables (requiring small time increment simulation) are liable to have a recurring effect with interspersed periods of calm (allowing relatively larger time increment simulation). If, for instance, an economy experiences a rapid burst of investment activity at time t the effects will be gradually absorbed in the economy until suddenly at time $t + \theta$, when an excessive amount of plant reaches its age of retirement, further repercussion will result.

An outline block diagram for the simulation process is given in fig. 15. This shows a number of independent subroutines (variously known in computer terminology as subprograms or procedures) central to which is the subroutine DERIVF. This sub-

FIGURE 15

Flow chart for the simulation of a differential-delay system



routine is the only point where the characteristics of the particular model enter and it represents the evaluation of the derivative function of the state variables from the system equations. This operation may itself call upon either of the subroutines INTERP and INTEGRAL which are concerned with delayed variables. An historical record is maintained of all variables appearing with time-delayed arguments (designated HISTV for the scalar variable V). The storage is in terms of a discrete tabulation at a predetermined time interval. The subroutine INTERP enables evaluation of $V(t - \theta)$ for any delay θ by cubic interpolation between the nearest four tabular points. Likewise subroutine INTEGRAL evaluates $\int_{t-\theta}^{t} V(t - \tau)d\tau$ by a simple trapezoidal algorithm. Each section of the simulation process is now discussed.

6.1.1 The Runge-Kutta Set-Up Process

Fig. 15 depicts three initialisation steps which are necessary before integration can commence. Firstly a time scale factor, SF is set which determines the interval of storage for the historical record and corresponds also to the output interval. This interval is designated as one simulation time unit and SF is the number of simulation time units per year. Next the initial conditions are set which involves not only values of the state variables at $t = 0$ but also, because of the delay factors, values of the state variable on the interval $(-\infty, 0)$; or in other words the historical records must also be initialised. Finally the initial integration step length, H is set although this value is not too critical as it is subsequently adapted in the simulation. The integration step length can never exceed one simulation time unit.

A fourth order predictor-corrector method is used in the main integration process of the simulation. The solution is thus to be advanced by extrapolation from the previous four solution points; in the usual notation y_1 is estimated using the values y_0, y_{-1}, y_{-2} and y_{-3} together with the corresponding derivative values dy_0, dy_{-1} and dy_{-2}. Thus in order to determine the <u>first</u> four solution points (three excluding the initial state at $t = 0$) an independent Runge-Kutta subroutine is used which is designated

RKUTTA(N, H, XO, YO, Y1),

where the parameters are as follows:

N the dimension of the state vector,

H the integration step length,

XO the current value of the independent variable,

YO the current value of the state vector,

and Y1 the calculated value of the state vector at XO + H.

The formula for the calculation of the new value of the state vector is the n-dimensional form of the standard fourth order equations

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{6.1}$$

where $k_1 = f(x_0, y_0)$,

$$k_2 = f(x_0 + \frac{h}{2}, \; y_0 + \frac{hk_1}{2}),$$

$$k_3 = f(x_0 + \frac{h}{2}, \; y_0 + \frac{hk_2}{2}),$$

$$k_4 = f(x_0 + h, \; y_0 + hk_3)$$

and $\frac{dy}{dx} = f(x, y)$.

The subroutine RKUTTA naturally requires specification of the previously mentioned subroutine DERIVF which is in the form

DERIVF(RT, Y, DY) .

where the parameters are:

RT the current value of real time,

Y the current value of the state vector,

DY the calculated value of the derivative vector.

These values at the first four solution points for the state vector and its derivative vector are then stored in arrays designated Y[I, J] and DY[I, J] respectively. I takes integral values between 1 and n for each component of the n-vector and J values between -3 and 0 to hold the four successive results. It is assumed that 3h is less than one simulation time unit at this stage so that the first output point has not been reached and the historical record is up to date. The main simulation is now entered (see fig. 15).

6.1.2 The Predictor-Corrector Integration

The following pair of fourth order formula are used in the main integration

$$y^P{}_1 = y_{-3} + \frac{4h}{3}(2dy_0 - dy_{-1} + 2dy_{-2}) \qquad (6.2)$$

PREDICTOR

and

$$y^C{}_1 = \frac{1}{8}(9y_0 - y_{-2}) + \frac{3h}{8}(dy^P{}_1 + 2dy_0 - dy_{-1}) \qquad (6.3)$$

CORRECTOR

The predictor formula is that of Milne whilst the corrector is due to Hamming and is known to possess satisfactory numerical stability properties (unlike the better known Simpson corrector - see Ralston[64] for example). Eqn. *(6.2)* is first applied to obtain an estimate for y_1 (we shall call this estimate $y^P{}_1$) whence an estimate for dy_1 is obtained. The latter is substituted in eqn. *(6.3)* to obtain $y^C{}_1$ and if necessary dy_1 can be re-estimated to obtain a second corrected value, $y^C{}_1$.

To obtain an estimate for the error in each of these fourth order formulae substitute $y = x^5$ where $x_1 = x_0 + h$, etc. and where without loss of generality it may be assumed that $x_0 = 0$. If ε is the single step error estimate then for the predictor formula

$$\begin{aligned}\varepsilon^P &= y^P{}_1 - h^5\\ &= -243h^5 + \frac{4h}{3}(0 - 5h^4 + 2 \,.\, 5 \,.\, 16h^4) - h^5\\ &= \frac{112}{3}h^5,\end{aligned}$$

and similarly for the corrector formula

$$\begin{aligned}\varepsilon^C &= y^C{}_1 - h^5\\ &= \frac{1}{8}(0 + 32h^5) + \frac{3h}{8}(5h^4 + 0 - 5h^4) - h^5\\ &= 3h^5.\end{aligned}$$

Subtracting the errors in the above equations

$$\varepsilon^P - \varepsilon^C = y^P{}_1 - y^C{}_1 = \frac{103}{3}h^5 = \frac{103}{9}\varepsilon^C,$$

that is

$$\varepsilon^C = \frac{9}{103}(y^P{}_1 - y^C{}_1). \qquad (6.4)$$

Thus the difference between the predicted and final and corrected values may be used as an estimate of the error in the final answer. This estimate is compared with a prescribed tolerance for acceptability, and additionally is used to examine the possibility of increasing the integration interval. Doubling the interval of integration will increase the error estimate by a factor of the order of 32. In practice doubling of the interval is permitted if the error estimate is less than the prescribed tolerance by a factor of at least 100.

In the present application the state vector is of course n-dimensional but the generalisation is trivial. The entire function is undertaken by a subroutine designated

MHAMMING(N, H, XO, Y, DY, Y1, DY1, TOL, TEST)

where the parameters are

N, H	as before
XO	the current value of the independent variable,
Y, DY	the arrays holding the previous four solution points for the state vector and its derivative as detailed in sec. 6.1.1,
Y1, DY1	the computed values of the state vector and its derivative at XO + H,
TOL	the required accuracy expressed as an integer number of significant figures,
TEST	the indicator of achieved accuracy; if TEST = 0 the computed values are unacceptable, if TEST ≥ 1 the values are acceptable and if TEST = 2 the

integration step length may be doubled.

6.1.3 Adapting the Step-Length

The value of the parameter TEST in subroutine MHAMMING indicates one of three possible courses of action (the reader is again referred to the simulation block diagram in fig. 15). If TEST = 0 the integration step length, h must be reduced (it is in fact halved) and the computed values at x_1 are rejected. However, in order to initiate the predictor-corrector process again with the halved step length it is necessary to compute the value of the state vector and its derivative at $x_{-1/2}$ and $x_{-3/2}$ which subsequent to adaption will become x_{-1} and x_{-3} respectively. Furthermore although x_0 and x_{-1} subsequent to adaption become x_0 and x_{-2} respectively it has been found more satisfactory to recompute the state vector and its derivative at these points also. Thus starting with the old value of the state vector at x_{-2}, the step length is halved and the point x_{-2} becomes x_{-4}. Subsequent values of the state vector and its derivative at x_{-3} through to x_0 are then computed using the Runge-Kutta set-up process and the main simulation is re-entered.

If TEST = 2 on exit from the MHAMMING subroutine the step length may be doubled subject to certain other criteria. Firstly it must be noted that in doubling h the ordinates $(x_{-6}, x_{-4}, x_{-2}, x_0)$ are transformed into the ordinates $(x_{-3}, x_{-2}, x_{-1}, x_0)$. Thus the values of the state vector at x_{-6} and x_{-4} in particular must be available. These values are therefore normally retained, but may not be available if the simulation has just commenced or if the step length has just previously been doubled. This is the first criterion.

The other criteria relate to output which, it will be recalled, occurs at one simulation time unit intervals. To monitor the subdivision of the output interval in terms of the integration step length and the time to the next output are two integer parameters

Q the number of times H divides into the output interval (one time unit),

M the number of integration steps to the next output point.

Doubling of the integration step length is not permitted to take place unless both $Q > 1$ and M is even. Q and M are of course modified when H is changed and M is reduced by one each time the predictor-corrector process successfully performs an integration.

If TEST = 1 following the MHAMMING subroutine no amendment to the integration step length is made.

6.1.4 Output and the Storage of Historic Values

An output point is reached when $M = 0$. At this point the current value of the state variable is output together with any other information that is required. In an economic model many of the original variables may be lost in the reduced form equations but their values must now be calculated. Similarly it may be of interest to investigate the actual mechanics of the integration so it is helpful to output the maximum and minimum values of Q since the last output point.

It is also at this point that the historical record is amended to accommodate the latest values. At the same time the oldest items of information are lost for in practice the record must be limited in length. If a new value of the variable Y is to be stored this is held under the designation HISTY[0] and the subroutine HISTØRY performs the remaining function. The record for each variable is held in the form of a *push-down* list which is accessible either directly or via one of the subroutines INTERP or INTEGRAL.

At the end of execution of the subroutine HISTØRY the value of M is reset to coincide with that of Q.

6.1.5 Advancing the Solution

The final step whether or not output has taken place is to advance the solution in anticipation of re-entry into subroutine MHAMMING. If no change in integration step length has taken place then a simple *push-down* operation is performed on the arrays Y[I, J] and DY[I, J] with the new values y_1 and dy_1 being substituted at what now becomes the point x_0. This operation is performed by the subroutine UPDATE which is without parameters. The exception to this case is where the integration step length has been doubled in which case an amended routine is invoked to accommodate the transformation referred to in sec. 6.1.3.

Thus the outline of a general procedure to solve a system of differential-delay equations has been given. The only sections relating specifically to the model are the subroutine DERIVF and the input/output routines, and thus the above discussion and the actual simulation programs listed in Appendix 2 are quite broadly applicable.

6.2 Implementation of the Vintage and Putty-Clay Model Simulations

Appendix 2 in fact contains listings of three differential-delay system simulations all with the characteristics described in sec. 6.1 but varying in respect of the model equations. The three programs are

SS3 Simulation of basic vintage model III,

SS3A Simulation of vintage model IIIA with population growth,

SS4 Simulation of basic putty-clay model IV.

Each program is written in ALGOL and has been implemented on the University of Warwick I.C.L. 4130 computer. All three programs have produced successful simulations and included in the documentation is a sample output from each. Before discussing these results however it is desirable to consider a number of factors pertaining to the setting-up of the particular economic models with which we are concerned.

6.2.1 Adaption of the Equations for the Vintage Model

In defining the subroutine DERIVF in program SS3 we use the formulation of the basic model represented by eqns. *(4.17)*. The state vector is $[n, v, \theta]'$ and the system is treated in this differential equation form except that where v appears on the right hand side of eqns. *(4.17)* its value is recalculated by integration of the historical record for n rather than using the value input to the subroutine. In this context it will be remembered that

$$v = \int_{t-\theta}^{t} n(\tau)\,d\tau. \qquad (4.15)$$

This modification considerably improves the stability of the numerical integration process by removing a source of discrepancy which can otherwise lead to the build-up of error. The v component of the state vector is thus effectively redundant although it is retained purely for reasons of convenience.

The same device is used in program SS3A where labour supply is variable and where the model differential equations are eqns. *(4.24)*, *(4.25)* and *(4.11)*. The relationship corresponding to eqn. *(4.15)* is

$$v = exp\{-\beta t\} \int_{t-\theta}^{t} n(\tau)\ exp\{\beta\tau\}d\tau \qquad (4.22)$$

and hence in this case an exponential discount factor is included in the historical record, HISTN. Program SS3A cannot be used for the constant labour force case ($\beta = 0$) as numerically indeterminate expressions will result if $\beta \to 0$.

6.2.2 Adaption of the Equations for the Putty-Clay Model

For the putty-clay model an historical record is maintained for both of the variables v and c which appear in the system equations *(4.42)* through *(4.45)*. The state vector adopted is $[cn,\ v,\ \theta]'$ and we use the derivative form of the first of eqns. *(4.42)* which is, after some simplification

$$\sigma\left[\frac{d}{dt}\ (cn)\ +\ \alpha cn\right] = \Bigg\{S_w c^b(t\ -\ \theta)\left[n\ -\ \dot{n}(t\ -\ \theta)(1\ -\ \frac{d\theta}{dt})\right]$$
$$-\ (S_p\ -\ S_w)\left[bc^{b-1}(t\ -\ \theta)c'(t\ -\ \theta)\ -\ \alpha c^b(t\ -\ \theta)\right](1\ -\ \frac{d\theta}{dt})v\Bigg\}$$
$$\times\ exp\{-\alpha\theta\}. \qquad (6.5)$$

The other system equations are the second of eqns. *(4.42)* namely

$$\left[b \frac{c'(t - \theta)}{c(t - \theta)} + \alpha\right](1 - \frac{d\theta}{dt}) = - \gamma + \rho / (1 - v), \tag{6.6}$$

and

$$\frac{dv}{dt} = n - n(t - \theta)(1 - \frac{d\theta}{dt}). \tag{6.7}$$

Eqn. *(6.7)* is of course the derivative form of our old equation

$$v = \int_{t-\theta}^{t} n(\tau)d\tau, \tag{4.15}$$

and the same device for v as was used in program SS3 is again employed. $c'(t - \theta)$ which appears in both eqns. *(6.5)* and *(6.6)* is also estimated using a simple difference approximation on the historical record of c, HISTC. In the case of the putty-clay model a problem arises in obtaining the value of n, or equivalently c as the first component of the state vector gives their product.

The subroutine DERIVF which evaluates the derivatives of the system equations must firstly obtain a value for c by solution of the algebraic non-linear equations *(4.43)*, *(4.44)* and *(4.45)* in the variables c, r, λ and ϕ. The variables λ and ϕ are first eliminated (in practice this is done numerically) to leave a pair of simultaneous equations in c and r - basically the eqns. *(4.43)*. This pair of equations is then solved by a two-dimensional root search process which is naturally extremely time consuming. A small economy is that because we are integrating with a relatively small step length the values of c and r obtained in the previous call on DERIVF may be used as a reasonable approximation on the next call. Thus the value of c is calculated and hence the derivatives are eventually determined.

6.2.3 Parameter Estimates

Although the nature of the present work does not warrant the use of sophisticated estimation techniques, data for the United Kingdom from the Central Statistical Office publication of *National Income and Expenditure*[10] is used as the basic source. To estimate the parameters γ and ρ however we refer back to the original estimates made by Phillips[55].

In its original form, as based upon data relating to the 52 year period from 1862 to 1913, the relationship between the rate of change in money wages, $\frac{1}{w}\frac{dw}{dt}$ and the level of unemployment, U/L was estimated by Phillips as

$$\frac{1}{w}\frac{dw}{dt} = -\,0{\cdot}9 + 9{\cdot}638\left(\frac{U}{L}\right)^{-1{\cdot}394}. \tag{6.8}$$

However one of several re-estimates made later by Lipsey[40] was

$$\frac{1}{w}\frac{dw}{dt} = -\,1{\cdot}42 + 7{\cdot}06\left(\frac{U}{L}\right)^{-1} + 2{\cdot}32\left(\frac{U}{L}\right)^{-2}, \tag{6.9}$$

and although the Lipsey data source for wage rates was different (and it was argued superior) the curves when compared were almost identical. For our present purposes, since only the first two terms on the right hand side of eqn. *(6.9)* are to be used, the Lipsey equation is economized. We estimate $(U/L)^{-2}$ by least squares using the frequency distribution of the original Phillips unemployment data to provide an appropriate weighting. The table below gives this distribution.

Unemployment %	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	over 9
Frequency (total 52)	1	5	10	11	4	5	7	4	3	2

The resultant estimate of $(U/L)^{-2}$ was found to be

$$\left(\frac{U}{L}\right)^{-2} = -\,0{\cdot}30_1 + 1{\cdot}39_5 \left(\frac{U}{L}\right)^{-1}, \tag{6.10}$$

whence the economized form of Lipsey's estimate is

$$\frac{1}{w}\frac{dw}{dt} = -\,2{\cdot}12 + 10{\cdot}3 \left(\frac{U}{L}\right)^{-1}. \tag{6.11}$$

The graphs of eqns. *(6.8)* and *(6.11)* are compared in fig. 16 and as can be seen are almost indistinguishable at least between 1% and 4% (which is the most likely operating area). Finally converting from percentages to pure fractions our estimates for γ and ρ are

$$\left.\begin{aligned} \gamma &= 0{\cdot}0212 \quad (\simeq 0{\cdot}02), \\ \rho &= 0{\cdot}00103 \ (\simeq 0{\cdot}001). \end{aligned}\right\} \tag{6.12}$$

Together with α these parameters determine the equilibrium level of unemployment, and to estimate the former we can simply examine GNP (table I of C.S.O.[10]) which has exhibited almost perfect exponential growth since about 1950. Thus to obtain a datum for α consider for example the average growth in GNP for the period 1960 to 1970 which in real terms was 3·24% or

$$\alpha = 0{\cdot}0324 \ (\simeq 0{\cdot}03). \tag{6.13}$$

Taking the rounded parameter values given in brackets in eqns. *(6.12)* and *(6.13)*, the equilibrium employment level as given by eqn. *(4.19)* is

$$\bar{v} = 1 - \frac{\rho}{\alpha + \gamma} = 0{\cdot}98,$$

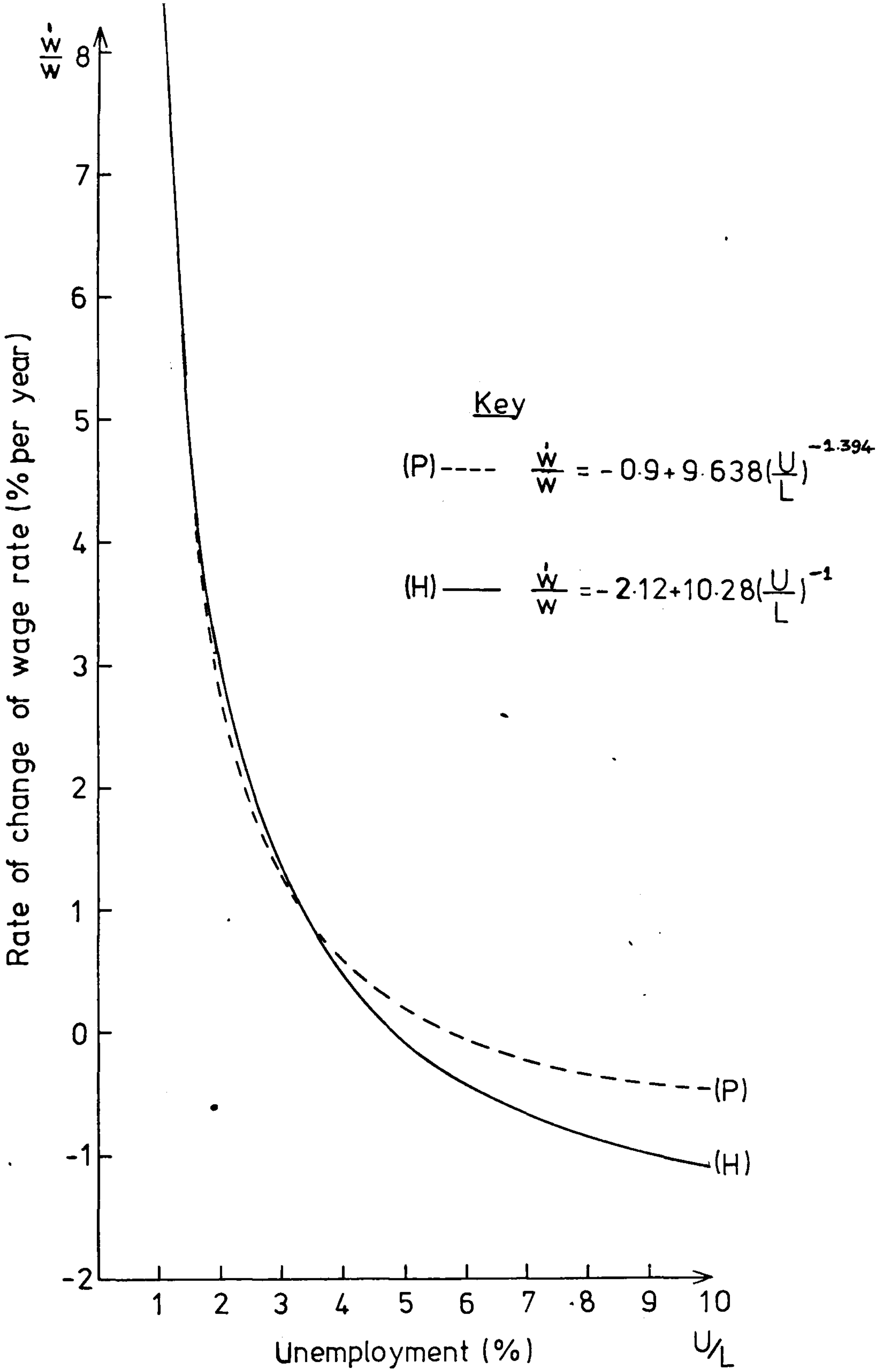


FIGURE 16 Approximation of Phillips curve

which represents unemployment at 2% or just less than ½ million for the U.K. economy (which does not seem unreasonable).

Estimates for the savings rates s_w and s_p are obtained from *National Income and Expenditure 1972*[10] in which a footnote to table 19 (income and expenditure account for the personal sector) gives personal saving as a percentage of disposable income for the period 1961-71 with values ranging from 7·6% to 8·9% with a mean of 8·2%. Reference to table 3 (corporate income appropriation account) of the same volume enables a similar calculation to be made of corporate saving as a percentage of after tax income. The relevant data is presented in fig. 17 with resultant values in the range 47% to 55% with a mean of 51%. Thus the values of the savings rates are

$$\left.\begin{aligned} s_w &= 0{\cdot}082 \quad (\simeq 0{\cdot}08) \\ s_p &= 0{\cdot}51 \quad (\simeq 0{\cdot}5). \end{aligned}\right\} \tag{6.14}$$

The estimation procedure for the constant capital-output ratio, σ in the vintage model is not altogether satisfactory for although table 64 of *National Income and Expenditure, 1972* gives values for gross capital stock at 1963 replacement cost, we have no means of determining the full capacity output potential of the capital. The best we can do is to ignore any under-utilisation of capacity, but for our purposes this will hardly matter. Gross domestic product at 1963 market prices is given in table 14 of *National Income and Expenditure, 1972*. The ratio of gross capital stock to gross domestic product for the period 1961-71 is then as in the following table.

Year	'61	'62	'63	'64	'65	'66	'67	'68	'69	'70	'71
Capital/Output Ratio	3·4	3·5	3·5	3·4	3·5	3·5	3·6	3·6	3·7	3·8	3·8

£ million	1961	1962	1963	1964	1965	1966	1967	1968	1969	1970	1971
Total income	6,252	6,408	7,090	7,892	8,481	8,301	8,872	10,059	10,605	11,112	12,357
less Taxes paid in U.K. and abroad	1,452	1,407	1,443	1,755	1,484	1,904	2,064	2,730	2,875	2,846	2,886
Net disposable income	4,800	5,001	5,647	6,137	6,997	6,397	6,808	7,329	7,730	8,366	9,471
Undistributed income (saving)	2,588	2,686	3,114	3,388	3,879	3,095	3,349	3,493	3,605	3,945	4,932
Saving as a % of disposable income	54	54	55	55	55	48	49	48	47	47	52

source: National Income and Expenditure 1972.

FIGURE 17 Corporate income appropriation account for U.K. 1961–1971

Thus in the vintage model we take

$$\sigma = 3{\cdot}6 \tag{6.15}$$

No value is estimated for the Cobb-Douglas parameter, b in the variable capital-output ratio putty-clay model but its value is generally assumed to lie in the range $0{\cdot}25$ to $0{\cdot}5$.

6.2.4 Input/Output Variables and Initial Conditions

The type of differential-delay model to be simulated can be regarded as an infinite dimensional system, and in prescribing initial conditions it is necessary in practice to set up a string of initial values for the delayed variable(s) on the interval $t = [-T, 0]$ as well as values at $t = 0$ for the remaining non-delayed dependent variables. T takes some value in excess of the largest delay θ that is likely to occur. However, this having been said the initial conditions, v_0, θ_0 and $n_0(\tau)$* where τ belongs to $[-T, 0]$ must be chosen to be consistent with the equation

$$v_0 = \int_{-\theta_0}^{0} n_0(\tau)\ \exp\{\beta\tau\}d\tau, \tag{6.16}$$

which is the reduction of eqn. *(4.22)* at $t = 0$. This applies to both the vintage and putty-clay models, where of course in the simulation $n_0(\tau)$ is held as a discrete function.

* *The zero subscript will be used to denote initial conditions which are at* $t = 0$, *and in particular* $n_0(\tau)$ *is the initial value of the time delayed variable* n *which is a function of* τ *for* $\tau \leq 0$. *For simplicity* $n_0(0)$ *is denoted just by* n_0.

What we have chosen to do for all presently reported simulation studies is to assume that $n_0(\tau) = \bar{n}$ for $\tau < 0$; that is we assume that the system was in equilibrium up until time $t = 0$ when a disturbance occurs. This disturbance is imposed upon the system by means of the initial values u_0 and v_0 representing the share of wages and employment respectively. For the vintage model u is given by

$$u = \frac{wE}{Y}, \tag{6.17}$$

but from eqns. *(4.7)* and *(4.8)*

$$\begin{aligned}\sigma Z &= (s_w - s_p)wE + s_p Y \\ &= \left[s_w - s_p(1 - \frac{1}{u})\right]wE.\end{aligned}$$

Substituting for Z and w from eqns. *(4.4)* and *(4.5)*

$$\sigma a_0 N \exp\{\alpha t\} = \left[s_w - s_p(1 - \frac{1}{u})\right]a_0 \exp\{\alpha(t - \theta)\}E;$$

that is

$$\sigma n = \left[s_w - s_p(1 - \frac{1}{u})\right] \exp\{- \alpha\theta\}v. \tag{6.18}$$

Hence given u_0 and v_0, n_0 and θ_0 can be determined between eqns. *(6.16)* and *(6.18)*. The actual evaluation of n_0 and θ_0 is carried out numerically as part of the initialisation process of the simulation.

Similarly for the putty-clay model substituting in eqn. *(4.8)*

$$I = \left[s_w - s_p(1 - \frac{1}{u})\right]wE,$$

but this time from eqn. *(4.1)*

$$w = \frac{Z(t - \theta)}{N(t - \theta)}$$

$$= \frac{1}{\sigma} I^b(t - \theta)N^{-b}(t - \theta) \exp\{\alpha(1 - b)(t - \theta)\}$$

by eqn. *(4.30)*. Substituting for $k = I/N = c \exp\{\alpha t\}$ we get

$$\sigma cN \exp\{\alpha t\} = \left[s_W - s_P(1 - \frac{1}{u})\right]c^b(t - \theta) \exp\{\alpha(t - \theta)\}E,$$

or

$$\sigma cn = \left[s_W - s_P(1 - \frac{1}{u})\right]c^b(t - \theta) \exp\{-\alpha\theta\}v. \tag{6.19}$$

So once again given u_0 and v_0 and assuming that $c_0(\tau) = \bar{c}$ for $\tau < 0$ the initial value of the state vector $[c_0n_0, v_0, \theta_0]'$ is determined by eqns. *(6.16)* and *(6.19)*, and the putty-clay simulation can proceed.

In both simulation programs SS3 and SS4 the variables u and v are included in the print list at each output point.

6.3 Results from simulation runs

A selection of results is illustrated below as representative of the characteristics displayed over many runs with various parameter settings for both the vintage and putty-clay models. The pattern that emerged was very similar even between the two basic models, except of course that the much more complex putty-clay simulation was considerably slower - by a factor of the order of 60 (for the vintage model on the Warwick University 4130 configuration approximately 20 years was simulated in a CPU time interval of one minute).

Fig. 18 (a) depicts a number of solution trajectories in the (u, v)-phase plane for the basic vintage model with constant labour supply. In this case the parameter values are exactly as derived in sec. 6.2.3, and the equilibrium point shown in the figure was as computed in the stability analysis program SA3A to give, in particular, an equilibrium unemployment level of 2%. For the initial conditions demonstrated $\bar{v}$ has been disturbed by up to $\pm\frac{1}{2}\%$, and the resulting trajectories have been allowed to run for 10 years. Time scale markings along the trajectories are in years.

First of all the similarity with the phase diagram trajectories for model IIA shown in fig. 7 is striking. The same rapid initial response is observed, so that after the first year the deviation of unemployment from its equilibrium value has been largely eliminated. This phase is followed by a slow correction of the wage share towards its equilibrium value, $\bar{u}$. The stability result predicted in sec. 5.2.2 is confirmed as shown more clearly in fig. 18 (b), where the simulation has been allowed to run for 50 years with identical parameter settings. Here the trajectory is clearly seen to cycle in towards the point of equilibrium growth.

Whilst the time scale of this second phase of the motion diminishes its practical importance, it is nevertheless interesting to observe the effect of the time delay in the system. Let $t = t_0$ at the start of a run and let the simulation time at subsequent *turning* points of the trajectory be t_1, t_2, t_3, etc. as shown in fig. 18 (b). We observe that at $t = t_2$ the life of plant variable (not shown in the phase diagram) is

$$\theta(t_2) = t_2 - t_0$$

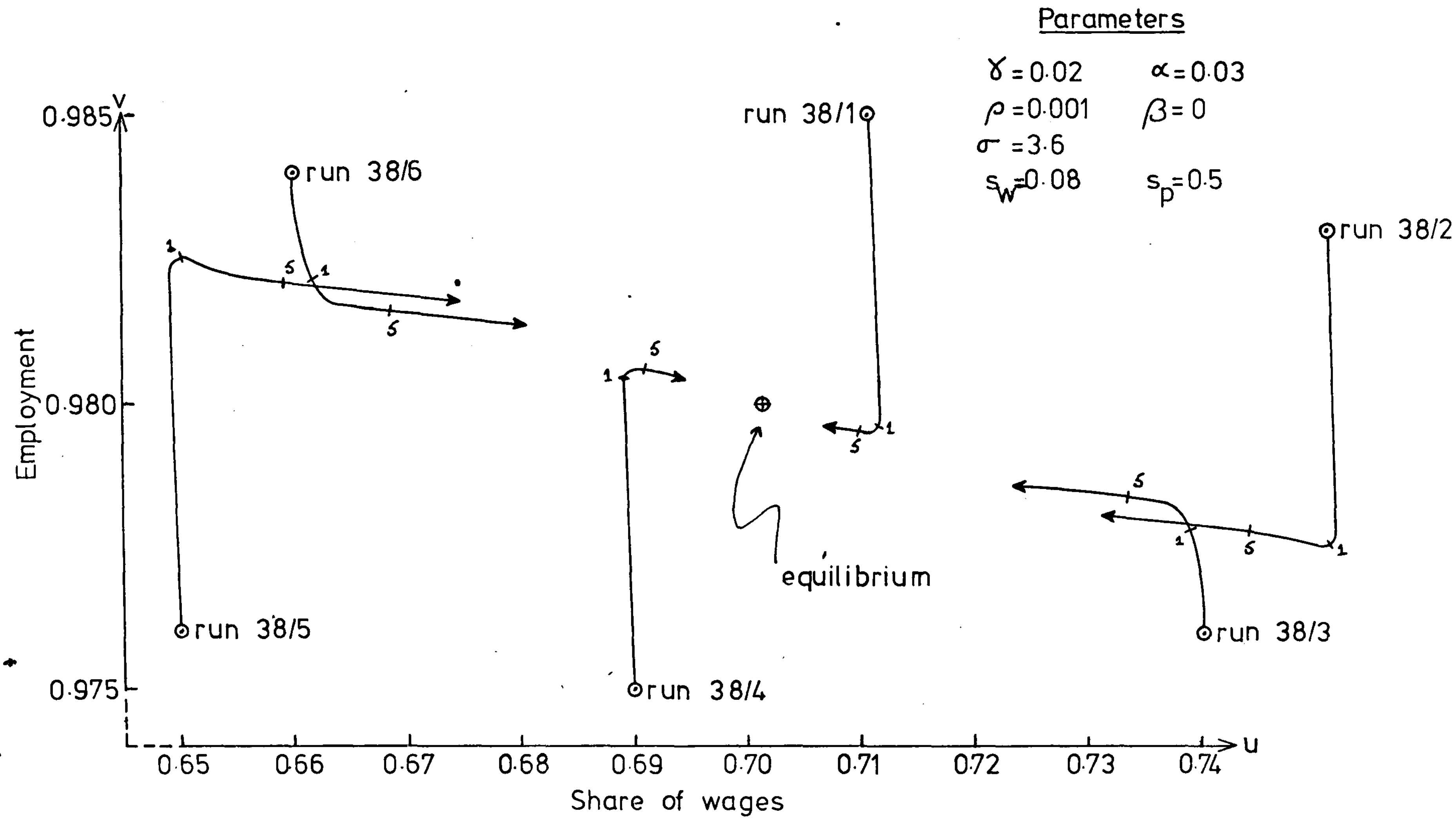


FIGURE 18(a) Solution trajectories for the basic Vintage model III

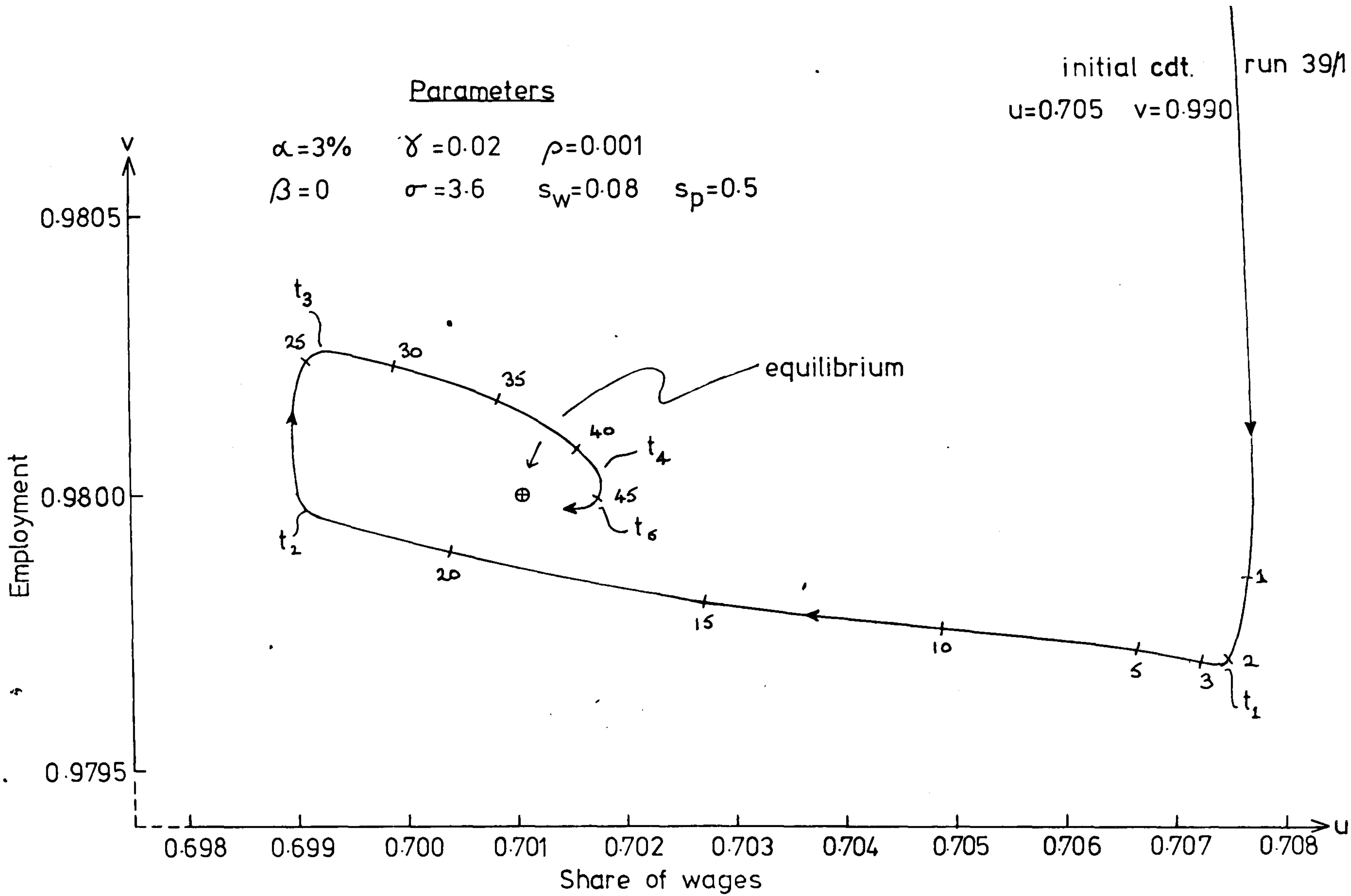


FIGURE 18(b) Solution trajectories for basic Vintage model III (50 year run) and similarly at t_3, t_4 and t_5 that

$$\theta(t_3) = t_3 - t_1,$$

$$\theta(t_4) = t_4 - t_2,$$

$$\theta(t_5) = t_5 - t_3, \text{ etc.}$$

The movement of the economy from t_0 to t_1 is perpetuated in a significant change in employment between t_2 and t_3, and then again between t_4 and t_5 and repeatedly until the effect is damped out. The mean period of the cyclical component of the motion is thus $2\bar{\theta}$.

Figs. 19 (a) and (b) show the same effect after a modification of parameter values. This is a classical savings case with a number of different length runs. Of course many of the trajectories intersect one another as for example runs 8/2 and 14/1 in fig. 19 (a) which again results from the time delay characteristics of the model.

Fig. 20 refers to the vintage model IIIA with population growth. We revert back to the original parameters of sec. 5.2.3 with labour supply now growing at 2% per annum. Again the equilibrium point is calculated by program SA3A and the simulation run using program SS3A which is also included in appendix 2. The latter program is a minor modification of that used to generate the trajectories in figs. 18 and 19. As can be seen however the effect of introducing growth in labour supply is very small compared with the earlier results. Again the stable equilibrium prediction of sec. 5.2.3 is confirmed.

The last case for the vintage model is that in which labour supply is decreasing, and the Nyquist analysis in sec. 5.2.2 has already shown that equilibrium growth is unstable in this situation. Fig. 21 shows a number of different trajectories computed with

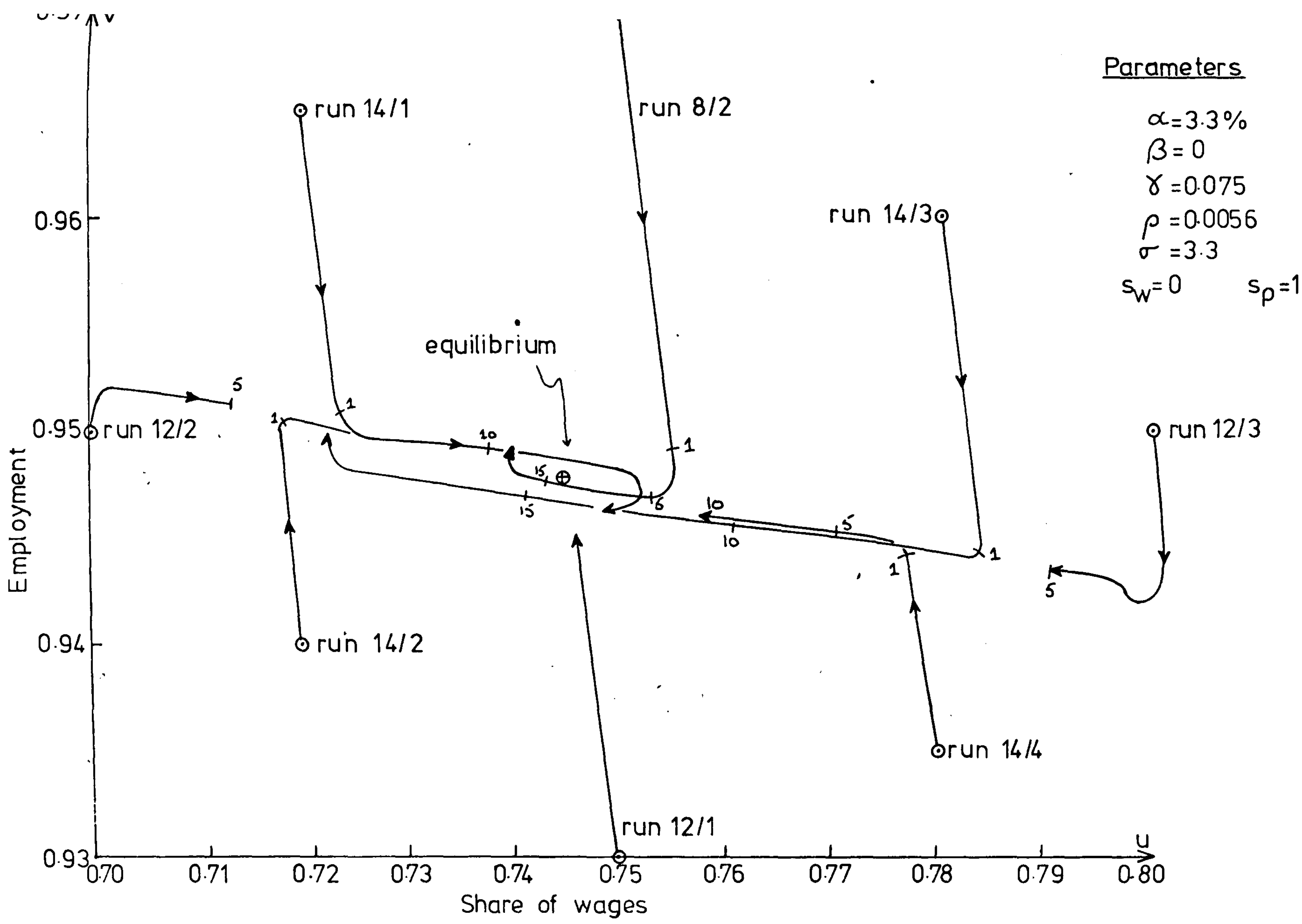


FIGURE 19(a) Solution trajectories for basic Vintage model III with classical savings

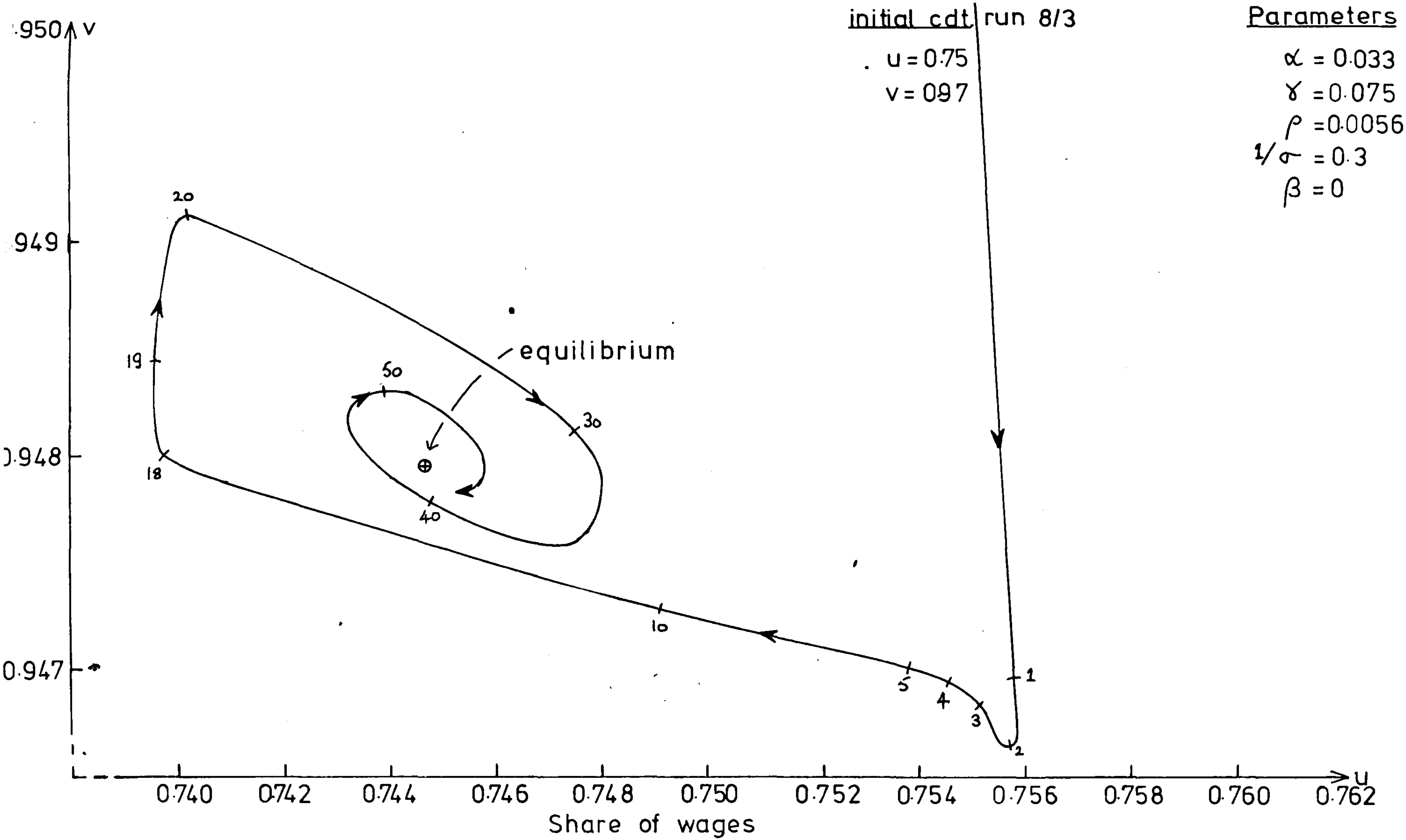


FIGURE 19(b) Solution trajectory for basic Vintage model III with classical savings (60 year run)

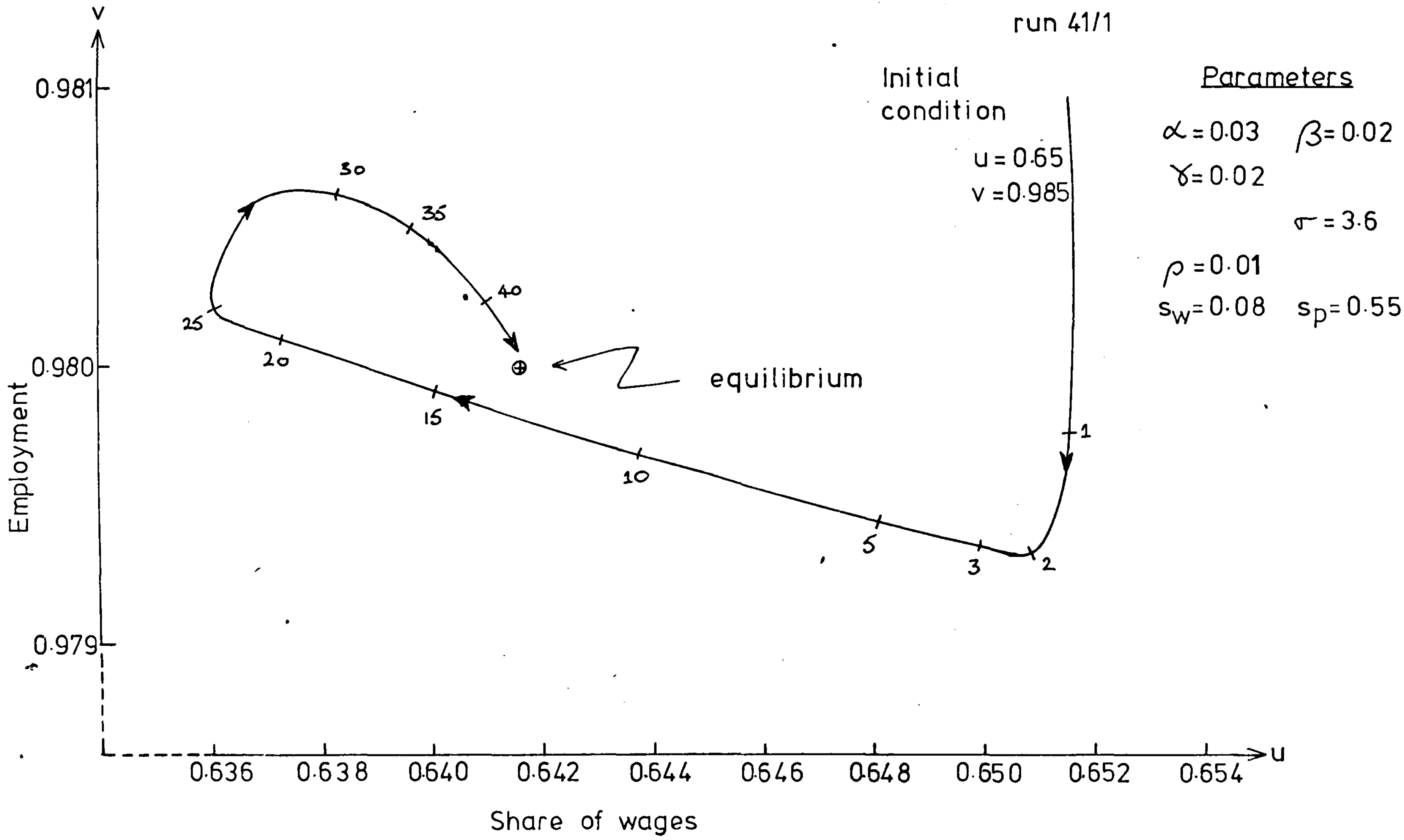


FIGURE 20 Solution trajectory for the Vintage model III A with growing labour supply ($\beta = +2\%$)

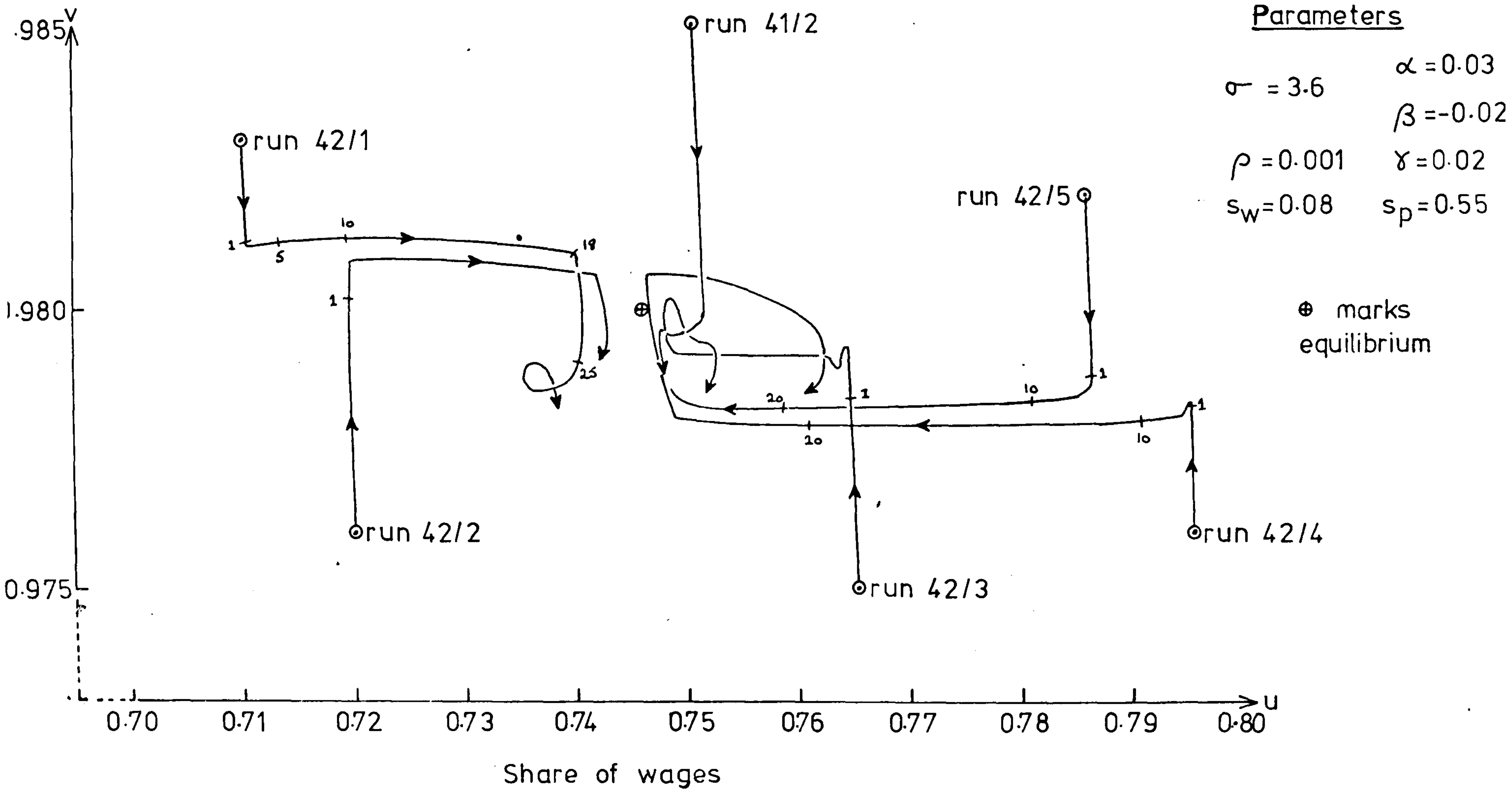


FIGURE 21 Solution trajectories for Vintage model IIIA with declining labour supply (β=-2%)

$\beta = -2\%$. Significantly the initial rapid response of the system in which the disturbance in unemployment is approximately corrected is still characteristic and this is a *stabilising response*. It is only after this initial phase that the system slowly drifts away from the point of equilibrium growth as disturbances in the time delayed variables begin to affect the system (remember that for $t < t_0$ the system was assumed to be in balanced growth). Thus the consequences for control are evidently not too gloomy.

Finally fig. 22 depicts trajectories from a number of runs of program SS4 which simulates the putty-clay model. The classical savings case is shown whilst the remaining parameters are similar to those previously used. We have constant labour supply and a value for the Cobb-Douglas index, b of 0·25. Earlier work in sec. 5.4 has shown the equilibrium point to be locally unstable for this particular case. The equilibrium is computed using program ES4 a listing of which can again be found in appendix 2. The outcome is only surprising in its similarity with the results discussed for the vintage model. In this case the initial response phase is even more rapid on account of the flexibility introduced into capital formation and the subsequent movement of the system relative to the equilibrium is again very slow. For example, run 21/2 in fig. 22 was a simulation of 10 years duration with the other runs all of about 5 years.

Further implications of these results will be raised in the concluding chapter which follows.

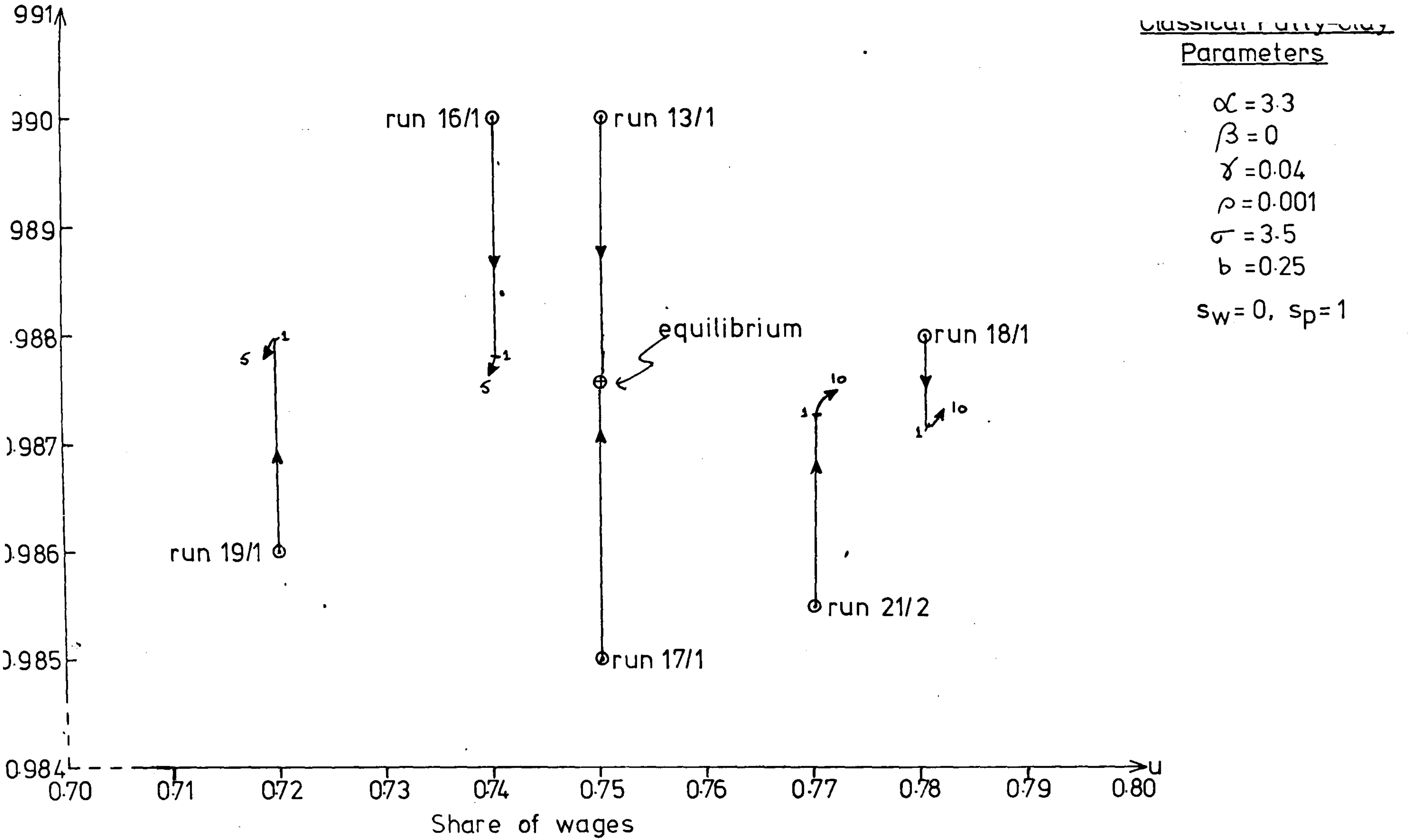


FIGURE 22 Solution trajectories for Putty-Clay model IV

CHAPTER 7

CONCLUDING REMARKS

The aims of this thesis have been twofold. Firstly it has been the intention to demonstrate the applicability to economic systems of a number of techniques from the field of control theory or more broadly from systems dynamics which hitherto have not been widely exploited. In this respect the present study owes its origin to the early work of Tustin[63] which has already been mentioned. Secondly, in terms of specific economic models, the chief aim has been to examine the effect of certain short/medium term disequilibrium adjustment mechanisms on situations of long term growth. This has lead to the consideration of models with variable time delay features; again an area that has so far received little attention in the literature but one in which much remains to be learnt.

In dealing with linear models the analysis has entailed identification of the system characteristic equation and subsequent location of the characteristic roots. (Such an approach is increasingly recognised as being important in many economic modelling exercises including econometric studies.) Further it has been shown in the case of the linear model IA how root locus plots can be usefully employed to study problems of system *sensitivity*. However, in addressing the questions of disequilibrium dynamics the concern has been with models that are non-linear, and a central issue has been that of equilibrium stability. Now whilst a linear model that is locally asymptotically stable is also asymptotically stable *in the large* this is not in general true for non-linear systems. The approach used has been to linearise the system in the neighbourhood of the equilibrium and apply the linear theory to obtain local stability results. This does not necessarily mean explicit evaluation of all of the roots of the characteristic equation, for using Nyquist analysis it has sometimes been possible to produce quite general (but not of course global) stability results. Whilst methods which attempt to give information on the stability of non-linear systems *in the large* as well as locally do exist* it has not been possible at the present time to obtain results of this nature for the current non-linear differential-delay models. In order to obtain more global information digital computer simulation studies were undertaken. Whilst obviously lacking the generality for which the analyst strives, the information which simulation can provide is invaluable practical evidence of how the system actually behaves under specific operating assumptions. However a great danger with the digital simulation of complex mathematical models is the ease with which erroneous results can be generated, and stringent precautions must be taken to guard against such a possibility.

Among the macro-models that have been analysed the linear models of chapter 2 are included by way of introduction, and whilst admittedly naive they nevertheless do have an instructive value. Such linear models are however limited in their ability to represent only the equilibrium dynamics of an economic system and as such are inadequate for our main purpose. The Goodwin model of chapter 3 is a non-linear disequilibrium model but at the same time has great simplistic appeal. The model demonstrates how endogenous cycles can be perpetuated under certain conditions, and as such could account

* *The second method of Liapunov (see Ogata[48]) is probably the best known of such methods.*

for the classic *trade* or *business* cycle observed in advanced economies. In modifying the Goodwin model to introduce, in particular, an approximation to finite plant life in model IIA, the oscillatory form of solution gave way to a strictly stable motion in which all paths converged in time towards the equilibrium. That is not to say that such a system would not oscillate in the presence of continued disturbances; and indeed Klein, who has considered the introduction of random shocks in *stochastic* linear and non-linear models[38], has suggested that the stochastic case of models such as IIA would in fact probably produce just such an oscillatory motion. This remains an area of current and future research.

Time delays were introduced into the models at this point with the adoption of standard vintage technology assumptions in chapter 4. The consequent increase in the mathematical complexity of the models was significant and in the case of the *putty-clay* investment model the cost was particularly high, even with a straight-forward Cobb-Douglas production function. The *vintage* model assumed a constant capital-output ratio whilst emphasising the dual effects on equilibrium growth of unemployment disequilibrium dynamics and a vintage technology. The equilibrium stability studies on both models in chapter 5 produced a number of useful results and a check on later simulation exercises. Variation of the labour supply parameter for the *vintage* model served to underline possible difficulties to be faced should the supply decrease rather than grow.

Digital simulation of the basic *vintage* model with constant labour supply illustrated characteristics common to all the models derived in chapter 4. The model was disturbed from equilibrium and reacted with a two-phased response in returning to the stable equilibrium growth condition. A *fast manifold* and a *slow manifold* were identified. The former represented a relatively fast approximate correction in the level of unemployment brought about by the disequilibrium dynamics in that sector of the model, whilst the second phase was one of slow cyclical convergence towards the equilibrium growth path. The time scale for the second phase was found to be closely related to the time delay feature of the model and the cycles had periods of the order of 20-50 years. Now Fisher* has said that cycles with such time scales are of no interest to the economist, and this author is bound to agree with the general spirit of such a statement; however it may be in special circumstances, where perhaps the phase plane plot of the cycle takes on an approximately rectangular profile, that two segments or sides of the cycle are traversed relatively quickly and the remaining two correspondingly slowly. Just such a situation was found to exist with the present time delay models as trajectories in the phase plane moved from the fast to the slow manifold and vice-versa. In such a case cycles even with periods of the order quoted above could be important although in the present case the cycles did prove to be damped and so their importance is proportionately diminished.

The difference in time scale between the two manifolds and the rapid initial phase of the response to a disturbance is particularly amplified in the case of the *putty-clay* model. The characteristic equation for the linearised version of the latter was of the form

$$p(s) + exp\{-s\bar{\theta}\}q(s) = 0$$

as given by eqn. *(5.23)*. Now if the dominant roots of this equation

* *Conversations on North American study tour (see Appendix 1).*

are very large (in modulus) as would be required to give a rapidly damped mode then such roots can be calculated approximately by assuming that the exponential term in the characteristic equation above is negligible, and thus by solving $p(s) = 0$. Since $p(s)$ is a cubic polynomial this is easily done and so, for example, with the parameter set

$$\alpha = 0{\cdot}03 \qquad \rho = 0{\cdot}001 \qquad s_w = 0{\cdot}0$$

$$\beta = 0{\cdot}0 \qquad \sigma = 3{\cdot}6 \qquad s_p = 1{\cdot}0$$

$$\gamma = 0{\cdot}02 \qquad b = 0{\cdot}25$$

the three roots are approximately $-9{\cdot}05$, $-0{\cdot}091$ and $0{\cdot}0065$. Thus the rapidly damped mode is approximately a function of $exp\{-9{\cdot}05t\}$ which has a half-life given by

$$log_e 2 \;/\; 9{\cdot}05 \text{ years} \simeq 1 \text{ month.}$$

This means that a disturbance will be half eliminated in about a month, which is roughly confirmed by the simulation studies. The fact that the equilibrium was actually unstable in the *putty-clay* case is probably not of great concern, although the comments made in connection with a stochastic version of model IIA are equally pertinent in the case of this and the other time delay models. The similarity between the first phase of the response of the time delay models and the earlier model IIA has already been indicated.

In conclusion it is, hopefully, already clear that inter-disciplinary work between control theory and economics has and will continue to contribute to economic theory. Further developments are difficult to predict but whatever new directions do emerge it is a safe assumption that non-linear and time delay factors will always be important considerations in the application of control theory or systems dynamics to economics.

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APPENDIX 1

RESEARCH PUBLICATIONS AND ASSOCIATED ACTIVITIES

1. June 1970: Study tour of North American universities (on Royal Society travel award) visiting:

   (i) Dept. of Economics,
   Massachusetts Institute of Technology.

   (ii) Institute for Quantitative Analysis of
   Social and Economic Policy,
   University of Toronto.

   (iii) Dept. of Electrical Engineering,
   University of Waterloo, Ontario.

   (iv) Georgia Institute of Technology,
   Atlanta, Georgia.

   (v) Wharton Business School,
   Pennsylvania State University.

2. Jan. 1971: Publication jointly with P. C. Parks of paper entitled *Studies in the modelling and control of a national economy*, Int. J. Systems Science, Vol. 1, No. 4, 311-322.

3. July 1972: Presentation of lecture entitled *Delay equations in economic modelling*, Symposium on Differential-Delay Processes,

Control Theory Centre, University of Warwick, (summary of proceedings published).

4. July 1973: Publication of paper entitled *On the dynamic analysis of a vintage model of the national economy*, Int. J. Systems Science, Vol. 4, No. 2, 227-241.

5. July 1973: Presentation of paper entitled *A differential-delay model of cyclical growth in a national economy*, I.F.A.C./I.F.O.R.S. international conference on Dynamic Modelling and Control of National Economies, University of Warwick, (proceedings published).

6. In preparation: Revised and extended version of paper no. 5 above - submission invited by editor of I.F.A.C. journal Automatica.

APPENDIX 2

DOCUMENTATION OF COMPUTER SIMULATION PROGRAMS

This appendix gives a program listing for each of the following five programs:

Each program is written in ALGOL (I.C.L. 4100 version). The ALGOL language was considered to be far superior for this purpose than other available languages (despite its unpopularity outside Europe). Special purpose simulation languages were not considered to be flexible enough to deal with all the complexities of the models.

The function of each program is briefly outlined and the potential user is told of the form in which data should be supplied - in each case an example set of data is given. A sample of the output produced by a computer run is also included following each program listing, and this output will be seen to correspond to the example data just mentioned. The form of the output is self-explanatory.

A2.1 Program SA3

'TITLE' Equilibrium and stability analysis of the vintage model with population growth.

'FUNCTION' To determine the coordinates of the unique equilibrium point of the model in question; to compute the coefficients of the characteristic equation and the Routh-Hurwitz matrix; finally to compute coordinates along the Nyquist path. The equilibrium is determined using a Newton-Raphson algorithm.

'USAGE' The user is required to supply the following items of data in a sequence as specified in the line of the program quoted:

(i) system parameters (line 13)

(ii) Nyquist path parameters (line 95) where the path is to be computed for $s = js_1$ in steps of js_2 to js_3 and again for $s = js_4$ in steps of js_5 to js_6.

'EXAMPLE DATA'

'Item (i)' 0·03 0 0·02 0·001 3·6 0 0·5 0·08

'Item (ii)' 0 0·005 0·095 0·10 0·01 0·20

```
** ICL 4130 DES2 SYSTEM  :  SLAVE BIG        :  CORE 70K  :  VOL  4

&JOB;ES/R073/A3;SA3A;

&ALGOL;
LIBRARY
ALGOL

&LIST;

PROGRAM SA3A  EQUILIBRIUM AND STABILITY ANALYSIS OF VINTAGE MODEL
              WITH POPULATION GROWTH                                    H J HOWAR

"BEGIN""REAL"ALPHA,BETA,GAMMA,RHO,DELTA,THETA,SIGMA,A,SW,SP,PHI,F,FD
        "REAL"C,V,W,BIGT,N,S,S1,S2,S3,S4,S5,S6;
        "REAL"X,Y,NR,DR,NI,DI,D,REAL,IMAG;
        "REAL"P0,P1,P2,P3,Q0,Q1,Q2,Q3;
        "REAL"A11,A12,A13,A14,A21,A22,A24,B11,B14,B24,C34;
        "REAL"U,C13;
        "REAL"A32;
        "SWITCH"SS:=L1;                "INTEGER"SIGN;

        "READ"ALPHA,BETA,GAMMA,RHO,SIGMA,A,SP,SW;
        "PRINT"''L10'PARAMETER VALUES OF SYSTEM ARE'L2'',
        'ALPHA=',ALIGNED(1,5),SAMELINE,ALPHA,
        ''L'BETA=',BETA,
        ''L'GAMMA=',GAMMA,
        ''L'RHO=',RHO,
        ''L'SIGMA=',SIGMA,
        ''L'COBB-DOUGLAS INDEX=',A,
        ''L'SAVINGS RATES,SW=',SW,' AND SP=',SP;

"COMMENT" CALCULATE EQUILIBRIUM SOLUTION FOR MODEL 4;

        BIGT:=20;
        V:=1-RHO/(ALPHA+GAMMA);
    L1: THETA:=ALPHA*BIGT;
        X:=EXP(THETA);
        "IF"ABS(BETA)<0,1+5"THEN"
        "BEGIN"F:=SP+THETA*(SP-SW)-(SP-SIGMA*ALPHA)*X;
               FD:=ALPHA*(SP-SW)-(SP-SIGMA*ALPHA)*ALPHA*X;
               Y:=1
           "END""ELSE""BEGIN"
        Y:=EXP(-BETA*BIGT);
        F:=SP*Y+(1+ALPHA/BETA)*(SP-SW)*(1-Y);
        F:=F-(SP-SIGMA*(ALPHA+BETA))*X;
        FD:=((1+ALPHA/BETA)*(SP-SW)-SP)*BETA*Y;
        FD:=FD-(SP-SIGMA*(ALPHA+BETA))*ALPHA*X;
        "END"OF BETA NE ZERO;
        BIGT:=BIGT-F/FD;
        "IF"ABS((BIGT-THETA/ALPHA)/BIGT)>0,1+5"THEN""GOTO"L1;
        "IF"ABS(BETA)<0,1+5"THEN"
        "BEGIN"N:=V/BIGT;
               U:=SP/(SIGMA*X/BIGT-SW+SP);
```

```
        "END""ELSE"
        "BEGIN"N:=BETA*V/(1-Y);
               U:=SP/(BETA*SIGMA*X/(1-Y)-SW+SP);
        "END";
        W:=C:=0;
        "PRINT"''L5'SYSTEM EQUILIBRIUM IS GIVEN BY'L2'',
        'EMPLOYMENT=',SAMELINE,FREEPOINT(5),V,
        ''L'LIFE OF PLANT=',BIGT,
        ''L'PROPORTION OF LABOUR EMPLOYED ON NEW PLANT=',N,
        ''L'WAGE CONSTANT=',W,
        ''L'SHARE OF WAGES=',U,
        ''L'PER CAPITA INVESTMENT CONSTANT=',C;

"COMMENT"  CALCULATE THE COEFFICIENTS OF CHARACTERISTIC EQUATION FOR
           MODEL 3A;

        A11:=(SP-(ALPHA+BETA)*SIGMA+(SW-SP)/X)*ALPHA/SIGMA;
        B11:=ALPHA/SIGMA*SW/X*Y;
        A32:=-RHO/(1-V)↑2;
        A12:=(SW-SP)*ALPHA*V-SW*Y*N;
        A12:=ALPHA↑2/SIGMA*(SW-SP)/X-A32/SIGMA/X*A12;
        A13:=-ALPHA↑2/SIGMA*(N+ALPHA*V)*(SW-SP)/X+(ALPHA+BETA)*N*B11;
        A22:=-ALPHA*BETA*N*A32*Y;
        P3:=-ALPHA↑3;
        P2:=ALPHA↑2*(A11+A22);
        P1:=ALPHA*(-A11*A22+ALPHA*BETA*Y*N*A32+ALPHA*A12);
        P0:=ALPHA*(-BETA*Y*N*A11+A13)*A32;
        Q3:=0;
        Q2:=-ALPHA↑2*B11;
        Q1:=ALPHA*(A22*B11-ALPHA*Y*A12);
        Q0:=ALPHA*Y*(BETA*N*B11-A13)*A32;

"COMMENT"  END OF PARTICULAR MODEL;

        "PRINT"''L5'CHARACTERISTIC EQUATION IS GIVEN BY'L'
        P(S)+EXP(-ST)*Q(S)=0'L2'WHERE COEFFICIENTS IN ASCENDING ORDER
        'L'',PREFIX(''S4''),SCALED(5),SAMELINE,
        ''L'FOR P(S)',P0,P1,P2,P3,
        ''L'FOR Q(S)',Q0,Q1,Q2,Q3;

"COMMENT"  CALCULATE ROUTH-HURWITZ MATRIX;

        SIGN:="IF"P3*P2<0"THEN"1"ELSE"0;
        X:=(P2*P1-P3*P0)/P2;
        "IF"P2*X<0"THEN"SIGN:=SIGN+1;
        "IF"X*P0<0"THEN"SIGN:=SIGN+1;
        "PRINT"''L5'NUMBER OF POLES IN R.H.P.=',SAMELINE,DIGITS(1),SI

"COMMENT"  COMPUTE NYQUIST PATH;

        "READ"S1,S2,S3,S4,S5,S6;
        "PRINT"''L5'NYQUIST PATH'L2''S12'S'S11'X(REAL)'S9'Y(IMAG)'L''
        "FOR"S:=S1"STEP"S2"UNTIL"S3,S4"STEP"S5"UNTIL"S6"DO"
        "BEGIN"NR:=Q0-Q2*S↑2;
               NI:=Q1*S-Q3*S↑3;
               DR:=P0-P2*S↑2;
               DI:=P1*S-P3*S↑3;
               D:=DR↑2+DI↑2;
               X:=(NI*DI+NR*DR)/D;
               Y:=(DR*NI-NR*DI)/D;
```

```
                     REAL:=X*COS(S*BIGT)+Y*SIN(S*BIGT);
                     IMAG:=Y*COS(S*BIGT)-X*SIN(S*BIGT);
                     "PRINT"ALIGNED(6,6),S,SAMELINE,REAL,IMAG;
               "END"OF LOOP S;
     "END"OF LOT;
  400   MC
  922  CODE
 1322 TOTAL


&RUN;
PROGRAMS
DRO
```

RESULTS

```
PARAMETER VALUES OF SYSTEM ARE

ALPHA= 0,03000
BETA= 0,00000
GAMMA= 0,02000
RHO= 0,00100
SIGMA= 3,00000
COBB-DOUGLAS INDEX= 0,00000
SAVINGS RATES,SW= 0,08000 AND SP= 0,50000



SYSTEM EQUILIBRIUM IS GIVEN BY

EMPLOYMENT= ,98000
LIFE OF PLANT= 23,744
PROPORTION OF LABOUR EMPLOYED ON NEW PLANT= ,04127
WAGE CONSTANT= ,00000
SHARE OF WAGES= ,68577
PER CAPITA INVESTMENT CONSTANT= ,00000



CHARACTERISTIC EQUATION IS GIVEN BY

      P(S)+EXP(-ST)*Q(S)=0

WHERE COEFFICIENTS IN ASCENDING ORDER ARE


FOR P(S)-3,0336₁₀-07-4,6352₁₀-08-9,1471₁₀-05-2,7000₁₀-05
FOR Q(S) 3,0336₁₀-07 3,8318₁₀-06-2,9430₁₀-07 0,0000₁₀+00
```

NUMBER OF POLES IN R.H.P.= 2

NYQUIST PATH

S	X(REAL)	Y(IMAG)
0.000000	-1.000000	0.000000
0.005000	-1.008021	0.056920
0.010000	-1.032719	0.117497
0.015000	-1.076150	0.185931
0.020000	-1.142353	0.267699
0.025000	-1.238581	0.370012
0.030000	-1.378030	0.508304
0.035000	-1.586308	0.703922
0.040000	-1.919142	1.007293
0.045000	-2.523037	1.544391
0.050000	-3.941903	2.759905
0.055000	-11.208500	8.127290
0.060000	9.139652	-11.397262
0.065000	2.898717	-3.735300
0.070000	1.555574	-2.328967
0.075000	0.965045	-1.732601
0.080000	0.629157	-1.397707
0.085000	0.409970	-1.179011
0.090000	0.254174	-1.021508
0.100000	0.045113	-0.800622
0.110000	-0.089190	-0.642330
0.120000	-0.180677	-0.514424
0.130000	-0.243093	-0.403502
0.140000	-0.283257	-0.303651
0.150000	-0.305052	-0.212428
0.160000	-0.311077	-0.129143
0.170000	-0.303413	-0.054010
0.180000	-0.283985	0.012327
0.190000	-0.254740	0.069093

STORE LEFT 60982 USED 5631

&END;
CPU TIME = 0000 13.632 REAL TIME 00 00 25
A

A2.2 Program SS3

'TITLE' Simulation of national economic system with basic vintage model.

'FUNCTION' To provide full scale simulation of system in question with variable initial conditions, output interval and run length. The method of solution, including integration of the system equations and maintenance of historical records, is discussed fully in ch.6.

'USAGE' The user is required to supply the following items of data in a sequence as specified in the line of the program quoted:

(i) system parameters (line 156)

(ii) coordinates of system equilibrium (line 169)

(iii) number of runs required

(iv) initial conditions, run length in years (LIFE) and scale factor (SF) for each run (line 172).

'EXAMPLE DATA'

'Item (i)' 0·03 0 0·02 0·001 0 0
0 0 0 0·08 0·55 3·6

'Item (ii)' 0·98000 22·428 0·04369 0 0

'Item (iii)' 1

'Item (iv)' 0·65 0·976 5 4

```
** ICL 4130 DES2 SYSTEM  :  SLAVE BIG        :  CORE 70K  :  VOL  4

&JOB;ES/R073/S3;SS3;



&ALGOL;
LIBRARY
ALGOL


&LINES;1000;


&LIST;

PROGRAM SS3   SIMULATION OF NATIONAL ECONOMIC SYSTEM WITH BASIC
              VINTAGE MODEL                                    M J HOWARTH;

"BEGIN""INTEGER"SIM,NSIM,LIFE,DIM,I,J,ST,Q,MAXQ,M,TEST;
        "BOOLEAN"DOUBLE;
        "REAL"ALPHA,BETA,GAMMA,RHO,DELTA,ETA,XI,ZETA,SIGMA,SW,SP;
        "REAL"V0,BIGT0,W0,K0,N0,U,V,A,BIGT,K,W,N,SF,H,PT;
        "REAL"THETA,R,C,LAMBDA,RST,C0,R0;
        "ARRAY"HISTN,HISTC[0:500];

"PROCEDURE"DERIVF(RT,Y,DY);
            "VALUE"RT;
            "REAL"RT;    "ARRAY"Y,DY;

"COMMENT"THIS PROCEDURE CALCULATES THE DERVATIVE OF THE STATE VECTOR,
          DY IS THE OUTPUT VARIABLE AND RT REPRESENTS REAL TIME;
       "BEGIN""INTEGER"I,J;      "REAL"AGE,P,Q,NAGE,X;   "ARRAY"FUN[-1

"REAL""PROCEDURE"INTERP(FUN,X);
                 "REAL"X;              "ARRAY"FUN;

"COMMENT" THIS PROCEDURE INTERPOLATES FOR FUN[X] IN THE ARRAY FUN[-1:
          USING AITKEN'S ALGORITM FOR ITERATED LAGRANGIAN INTERPOLAT

       "BEGIN""INTEGER"I,J;  "ARRAY"XX[-1:2],Y[-1:2,0:3];
             "FOR"I:=-1"STEP"1"UNTIL"2"DO"
             "BEGIN"XX[I]:=X-I;
                    Y[I,0]:=FUN[I];
                    "FOR"J:=1"STEP"1"UNTIL"I+1"DO"
                    Y[I,J]:=(XX[J-2]*Y[I,J-1]-XX[I]*Y[J-2,J-1])/(I-J
             "END"OF LOOP I;
             INTERP:=Y[2,3];
       "END"OF PROCEDURE INTERP;

"REAL""PROCEDURE"INTEGRAL(FUN);
                 "ARRAY"FUN;

"COMMENT" THIS PROCEDURE EVALUATES THE INTEGRAL OF FUN[0:500] BETWEEN
          (T-BIGT) AND T USING THE TRAPEZIUM RULE - GLOBAL VALUES OF
          P, Q AND SF ARE REQUIRED;
```

```
        "BEGIN""INTEGER"J;     "REAL"SUM;
                SUM:=(FUN[1]+FUN[I+1])/2;
                "FOR"J:=2"STEP"1"UNTIL"I"DO"SUM:=SUM+FUN[J];
                SUM:=SUM+(FUN[1]+FUN[0])*P/2;
                SUM:=SUM-Q/2*(FUN[I]*Q+FUN[I+1]*(2-Q));
                INTEGRAL:=SUM/SF;
        "END"OF PROCEDURE INTEGRAL;

    "COMMENT"       MODEL 3 - BASIC VINTAGE MODEL IN WHICH:
                                                      Y[1] IS N
                                                      Y[2] IS V AND
                                                      Y[3] IS BIGT;
               RST:=RT*SF;
               AGE:=1+SF*Y[3]+ENTIER(RST-0,1*6)-RST;
               I:=ENTIER(AGE);
               P:=RST-ENTIER(RST-0,1*6);
               Q:=I+1-AGE;
               HISTN[0]:=Y[1];
               V:=INTEGRAL(HISTN);
               "FOR"J:=-1,0,1,2"DO"FUN[J]:=HISTN[I+1-J];
               NAGE:=INTERP(FUN,Q);
               X:=-GAMMA*RHO/(1-V);
               DY[1]:=(Y[1]*(SP-ALPHA*SIGMA)-EXP(-ALPHA*Y[3])*((SP-SW)
               *(Y[1]+V*X)+X*SW*NAGE/ALPHA))/SIGMA;
               X:=X/ALPHA;
               DY[2]:=Y[1]-NAGE*X+V-Y[2];
               DY[3]:=1-X;
        "END"OF PROCEDURE DERIVF;

   "PROCEDURE"RKUTTA(N,H,X0,Y0,Y1);
             "VALUE"X0,Y0,H;
             "INTEGER"N;      "REAL"X0,H;     "ARRAY"Y0,Y1;

   "COMMENT" THIS PROCEDURE INTEGRATES THE N DIMENSIONAL SYSTEM OF EQUA
             DY/DX=F(X,Y)OVER ONE STEP OF LENGTH H FROM (X0,Y0) USING A
             ORDER RUNGE-KUTTA ALGORITHM,  OUTPUT IS VIA THE ARRAY Y1 -
             PROCEDURE DERIVF MUST BE DECLARED GLOBALLY;

        "BEGIN""INTEGER"I,J;
               "ARRAY"Y,DY[1:N],K[1:N,1:3];
               DERIVF(X0,Y0,DY);
               "FOR"I:=1"STEP"1"UNTIL"N"DO"
               "BEGIN"K[I,1]:=DY[I];
                      Y[I]:=Y0[I]+H/2*DY[I];
               "END"OF LOOP I;
               DERIVF(X0+H/2,Y,DY);
               "FOR"I:=1"STEP"1"UNTIL"N"DO"
               "BEGIN"K[I,2]:=DY[I];
                      Y[I]:=Y0[I]+H/2*DY[I];
               "END"OF LOOP I;
               DERIVF(X0+H/2,Y,DY);
               "FOR"I:=1"STEP"1"UNTIL"N"DO"
               "BEGIN"K[I,3]:=DY[I];
                      Y[I]:=Y0[I]+H*DY[I];
               "END"OF LOOP I;
               DERIVF(X0+H,Y,DY);
               "FOR"I:=1"STEP"1"UNTIL"N"DO"Y1[I]:=Y0[I]+H/6*(K[I,1]+2*
               2]+2*K[I,3]+DY[I]);
        "END"OF PROCEDURE RKUTTA;
```

```
"PROCEDURE""MHAMMING(N,H,X0,Y0,DY0,Y1,DY1,TOL,TEST);
            "VALUE"H,X0;
            "INTEGER"N,TOL,TEST;  "REAL"H,X0;  "ARRAY"Y0,DY0,Y1,DY1;

"COMMENT" THIS PROCEDURE PERFORMS THE SAME FUNCTION AS RKUTTA BUT USI
          THE 4TH ORDER MILNE-HAMMING PREDICTOR-CORRECTOR METHOD.  IF
          THE OUTPUT PARAMETER TEST IS 0 THEN THE RESULTS CONTAIN AN
          ERROR > 0.1*TOL, IF TEST >0 THE RESULTS ARE ACCEPTABLE AND
          TEST =2 THE STEP LENGTH MAY BE DOUBLED;

    "BEGIN""INTEGER"I,J;          "ARRAY"PRED,CORR,L,K[1:N];
         "SWITCH"SS:=S1,S2;
         "FOR"I:=1"STEP"1"UNTIL"N"DO"CORR[I]:=PRED[I]:=Y0[I,-3]+4
         3*(2*DY0[I,0]-DY0[I,-1]+2*DY0[I,-2]);
  S1;    DERIVF(X0+H,CORR,DY1);
         "FOR"I:=1"STEP"1"UNTIL"N"DO"Y1[I]:=(9*Y0[I,0]-Y0[I,-2]+3
         (DY1[I]+2*DY0[I,0]-DY0[I,-1]))/8;
         "FOR"I:=1"STEP"1"UNTIL"N"DO""IF"ABS((Y1[I]-CORR[I])/CORR
         )>0.1*TOL/2"AND"ABS(Y1[I]-CORR[I])>0.1*TOL/20"THEN"
         "BEGIN""FOR"I:=1"STEP"1"UNTIL"N"DO"CORR[I]:=Y1[I];
              "GOTO"S1;
         "END"OF CONDITION THEN;
         TEST:=2;
         "FOR"I:=1"STEP"1"UNTIL"N"DO"
         "BEGIN"L[I]:=ABS(Y1[I]-PRED[I]);
              K[I]:=ABS(L[I]/PRED[I]);
              "IF"K[I]>5*0.1*TOL"AND"L[I]>0.1*TOL/2"THEN"
              "BEGIN"TEST:=0;
                   "GOTO"S2;
              "END"OF CONDITION THEN;
              "IF"K[I]>0.1*TOL/20"AND"L[I]>0.1*TOL/200"THEN"
              TEST:=1;
         "END"OF LOOP I;
  S2;"END"OF PROCEDURE MHAMMING;

"PROCEDURE"HISTORY;

"COMMENT" THIS PROCEDURE UPDATES THE HISTORICAL TIME PATH OF VARIABLE
          SUBJECT TO A TIME DELAY.  THE HISTORICAL RECORD IS STORED A
          INTERVALS OF ONE SIMULATION TIME UNIT AND HIST[0] HOLDS THE
          CURRENT VALUE OF THE VARIABLES;

    "BEGIN""INTEGER"I;
         "FOR"I:=500"STEP"-1"UNTIL"1"DO"
         "BEGIN"HISTM[I]:=HISTM[I-1];
              HISTC[I]:=HISTC[I-1];
         "END"OF LOOP I;
    "END"OF PROCEDURE HISTORY;

"COMMENT" INPUT SYSTEM PARAMETERS AND EQUILIBRIUM VALUES FOLLOWED BY
          DETERMINANTS OF THE CURRENT SIMULATION RUNS.  THE SCALE FA
          SF, IS THE NUMBER OF SIMULATION TIME UNITS PER YEAR AND
          CORRESPONDS TO THE FINAL STORAGE AND OUTPUT INTERVAL;

    "READ"ALPHA,BETA,GAMMA,RHO,DELTA,XI,ETA,ZETA,A,SM,SP,SIGMA;
    "PRINT"''L2'PARAMETER VALUES OF SYSTEM ARE'L'',
    ''L'ALPHA=',ALIGNED(1,5),SAMELINE,ALPHA,
    ''L'BETA=',BETA,
    ''L'GAMMA=',GAMMA,
```

```
        ''L'RHO=',RHO,
        ''L'SIGMA=',SIGMA,
        ''L'DELTA=',DELTA,
        ''L'XI=',XI,
        ''L'ETA=',ETA,
        ''L'ZETA=',ZETA,
        ''L'COBB-DOUGLAS INDEX=',A,
        ''L'SAVINGS RATES,SW=',SW,'AND SP=',SP,''L'';
        "READ"V0,BIGT0,N0,C0,R0;
        "READ"NSIM;
        "FOR"SIM:=1"STEP"1"UNTIL"NSIM"DO"
        "BEGIN""READ"U,V,LIFE,SF;
              "FOR"I:=0"STEP"1"UNTIL"500"DO"HISTN[I]:=N0;
              RT:=0;
              C:=C0;
              "PRINT"''L5'RUN',SAMELINE,DIGITS(1),SIM,''L2'U=',
              ALIGNED(1,3),U,' AND V=',V,' INITIALLY'L2'SIMULATE',
              DIGITS(2),LIFE,' YEARS WITH OUTPUT INTERVAL OF',
              ALIGNED(2,0),12/SF,' MONTHS';
              "PRINT"''L10'TIME IN'S6'SHAPE OF'S6'RELATIVE'S7'PLANT'S7
              'EMPLOYMENT'S5'INTEGRATION'L' YEARS'S8'WAGES'S7'EMPLOYME
              ,''S6'LIFE'S7'ON NEW PLANT.    SUBDIVISION';

"COMMENT" ENTER INITIAL SET-UP PROCESS FOR PREDICTOR-CORRECTOR METHOD

"COMMENT" SET UP MODEL 3 - BASIC VINTAGE MODEL;

              DIM:=3;
              "BEGIN""ARRAY"Y,DY[1:3,-3:0],Y0,Y1,DY0,DY1,VAL,DVAL,OLD4
                    OLD5[1:3];      "SWITCH"SS:=L1,L2;
                    "REAL"IA,IB,IC,OLDT;

"PROCEDURE"UPDATE;

"COMMENT" THIS PROCEDURE UPDATES THE IMMEDIATE PAST RECORD OF THE STA
        VARIABLE TO PERMIT A FURTHER APPLICATION OF MHAMMING;

        "BEGIN""INTEGER"I,J;
              "FOR"I:=1"STEP"1"UNTIL"DIM"DO"
              "BEGIN"OLD5[I]:=OLD4[I];
                    OLD4[I]:=Y[I,-3];
                    "FOR"J:=-3,-2,-1"DO"
                    "BEGIN"Y[I,J]:=Y[I,J+1];
                          DY[I,J]:=DY[I,J+1];
                    "END"OF LOOP J;
                    Y[I,0]:=Y1[I];
                    DY[I,0]:=DY1[I];
              "END"OF LOOP I;
        "END"OF PROCEDURE UPDATE;

                    Q:=64;
                    IA:=(2*SF*V+N0)*SIGMA;
                    IB:=2*SIGMA*N0*SF;
                    IC:=V*(SP*(1/U-1)+SW);
                    IA:=BIGT:=IA/IB;
                    IC:=IC/IB;
     L2:            OLDT:=BIGT;
                    BIGT:=IA-IC/EXP(ALPHA*OLDT);
                    "IF"ABS(OLDT-BIGT)>0,1+4"THEN""GOTO"L2;
                    N:=2*SF*(V-BIGT*N0)+N0;
```

```
                        Y[1,-3]:=Y0[1]:=U;
                        Y[2,-3]:=Y0[2]:=V;
                        Y[3,-3]:=Y0[3]:=BIGT;

"COMMENT" END OF PARTICULAR MODEL;

                        H:=1/Q/SF;
                        "PRINT"''L'',ALIGNED(3,2),RT,SAMELINE,ALIGNED(6,5
                        U,ALIGNED(7,5),V,ALIGNED(8,4),BIGT,ALIGNED(7,5),'
                        ''S4'',DIGITS(3),Q;
                        "FOR"J:=-2,-1,0"DO"
                        "BEGIN"RKUTTA(DIM,H,(2+J)*H,Y0,VAL);
                                DERIVF((3+J)*H,VAL,DVAL);
                                "FOR"I:=1"STEP"1"UNTIL"DIM"DO"
                                "BEGIN"Y[I,J]:=Y0[I]:=VAL[I];
                                        DY[I,J]:=DVAL[I];
                                "END"OF LOOP I;
CARE
                        "END"OF LOOP J AND INITIAL SET-UP PROCESS;

"COMMENT" ENTER MAIN SIMULATION;

                        OLD4[3]:=OLD5[3]:=0;
                        MAXQ:=0;
                        "FOR"ST:=1"STEP"1"UNTIL"LIFE*SF+0,1"DO"
                        "BEGIN""IF"ST=1"THEN"M:=Q-3"ELSE"M:=0;
                                RT:=ST/SF-M*H;
    L1:                         HHAMMING(DIM,H,RT,Y,DY,Y1,DY1,4,TEST);

"COMMENT" TEST TRUNCATION ERROR AND IF UNACCEPTABLE REDUCE STEP LENGT
          OF INTEGRATION;

                                "IF"TEST=0"THEN"
                                "BEGIN"H:=H/2;
                                        M:=M*2;
                                        Q:=Q*2;
                                        "IF"Q>MAXQ"THEN"MAXQ:=Q;

"COMMENT" RE-ENTER A SET-UP PROCESS FOR PREDICTOR-CORRECTOR METHOD
          HAVING HALVED THE STEP LENGTH OF INTEGRATION;

                                        "FOR"J:=1"STEP"1"UNTIL"DIM"DO"
                                        "BEGIN"OLD4[J]:=Y[J,-2];
                                                OLD5[J]:=0;
                                                VAL[J]:=OLD4[J];
                                        "END"OF LOOP J;
                                        "FOR"I:=-3,-2,-1,0"DO"
                                        "BEGIN"RKUTTA(DIM,H,RT+I*H-H,VAL,Y1
                                                DERIVF(RT+I*H,Y1,DY1);
                                                "FOR"J:=1"STEP"1"UNTIL"DIM"D
                                                "BEGIN"Y[J,I]:=VAL[J]:=Y1[J
                                                        DY[J,I]:=DY1[J];
                                                "END"OF LOOP J;
                                        "END"OF LOOP I;
                                        "GOTO"L1;
                                "END"OF TEST FAIL;
                                M:=M-1;
                                RT:=RT+H;

"COMMENT" DOUBLE INTEGRATION STEP LENGTH IF APPROPRIATE;
```

```

                                DOUBLE:="FALSE";
                                "IF"TEST=2"AND"OLD5[3]"NE"0"AND"M=2*(M"DI
                                "AND"Q>1
                                "THEN"
                                "BEGIN"H:=H*2;
                                     M:=M"DIV"2;
                                     Q:=Q"DIV"2;
                                     DOUBLE:="TRUE";
                                "END"OFTEST DOUBLE;

    "COMMENT" TEST FOR OUTPUT POINT;

    "COMMENT" OUTPUT ROUTINE FOR MODEL 4 - BASIC VINTAGE MODEL WITH RETA

                                "IF"M=0"THEN"
                                "BEGIN"HISTU[0]:=U:=Y1[1];
                                     V:=Y1[2];
                                     BIGT:=Y1[3];
                                     U:=SP/(SIGMA*N*EXP(ALPHA*BIGT
                                     )/V-SW+SP);
                                     "PRINT"ALIGNED(3,2),RT,SAMELINE,
                                     ALIGNED(6,5),U,ALIGNED(7,5),V,ALIG
                                     (8,4),BIGT,ALIGNED(7,5),N,''S4'',
                                     DIGITS(3),MAXQ;

    "COMMENT" END OF PARTICULAR MODEL;

                                     HISTORY;
                                     MAXQ:=0;
                                "END"OF OUTPUT;

    "COMMENT" ADVANCE SOLUTION PRIOR TO RE-ENTERING PREDICTOR-CORRECTOR
            INTEGRATION ROUTINE;
                                "IF""NOT"DOUBLE"THEN"UPDATE"ELSE"
                                "FOR"J:=1"STEP"1"UNTIL"DIM"DO"
                                "BEGIN"Y[J,0]:=Y1[J];
                                     Y[J,-2]:=Y[J,-3];
                                     Y[J,-3]:=OLD5[J];
                                     OLD4[J]:=OLD5[J]:=0;
                                     DY[J,0]:=DY1[J];
                                     DY[J,-2]:=DY[J,-3];
CARE
                                "END"OF DOUBLE=TRUE;
                                "IF"M"NE"0"THEN""GOTO"L1;
                           "END"OF LOOP ST AND CURRENT RUN;
                      "END"OF ARRAY DECLARATION BLOCK;
                "END"OF LOOP SIM;
    "END"OF LOT;
   734   MC
  2412  CODE
  3146 TOTAL


 &RUN;
PROGRAMS
DRO
```

RESULTS

PARAMETER VALUES OF SYSTEM ARE

ALPHA= 0,03000
BETA= 0,00000
GAMMA= 0,02000
RHO= 0,00100
SIGMA= 3,60000
DELTA= 0,00000
XI= 0,00000
ETA= 0,00000
ZETA= 0,00000
COBB-DOUGLAS INDEX= 0,00000
SAVINGS RATES,SW= 0,08000AND SP= 0,55000

RUN 1

U= 0,650 AND V= 0,976 INITIALLY

SIMULATE 5 YEARS WITH OUTPUT INTERVAL OF 3 MONTHS

TIME IN YEARS	SHARE OF WAGES	RELATIVE EMPLOYMENT	PLANT LIFE	EMPLOYMENT ON NEW PLANT	INTEGRAT SUBDIVIS
0,00	0,65000	0,97600	22,3148	0,05221	64
0,25	0,64936	0,98023	22,3702	0,05247	64
0,50	0,64919	0,98219	22,3736	0,05260	4
0,75	0,64937	0,98279	22,3432	0,05265	4
1,00	0,64972	0,98288	22,2992	0,05266	4
1,25	0,65013	0,98279	22,2501	0,05265	4
1,50	0,65059	0,98268	22,1997	0,05264	4
1,75	0,65107	0,98258	22,1488	0,05263	4
2,00	0,65150	0,98249	22,0976	0,05261	4
2,25	0,65210	0,98242	22,0467	0,05260	4
2,50	0,65264	0,98237	21,9957	0,05258	4
2,75	0,65320	0,98233	21,9451	0,05255	4
3,00	0,65378	0,98229	21,8942	0,05252	4
3,25	0,65436	0,98226	21,8436	0,05250	4
3,50	0,65497	0,98222	21,7930	0,05246	4
3,75	0,65558	0,98220	21,7427	0,05243	4
4,00	0,65621	0,98219	21,6928	0,05239	4
4,25	0,65683	0,98216	21,6426	0,05235	4
4,50	0,65750	0,98215	21,5931	0,05231	4
4,75	0,65817	0,98212	21,5435	0,05227	4
5,00	0,55884	0,98212	21,4945	0,05222	4

A2.3 Program SS3A

´TITLE` Simulation of national economic system with vintage model and population growth.

´FUNCTION` As for program SS3 but catering exclusively for case of non-zero growth in labour supply.

´USAGE` The user is required to supply the following items of data in a sequence as specified in the line of the program quoted:

(i) system parameters (line 158)

(ii) coordinates of system equilibrium (line 171)

(iii) number of runs required

(iv) initial conditions, run length in years (LIFE) and scale factor (SF) for each run (line 174).

´EXAMPLE DATA`

´Item (i)` 0·03 --·02 0·02 0·001 0 0
0 0 0 0·08 0·55 3·6

´Item (ii)` 0·98000 19·895 0·04011 0 0

´Item (iii)` 1

´Item (iv)` 0·780 0·985 2 6

** ICL 4130 DES2 SYSTEM ; SLAVE BIG ; CORE 70K ; VOL 4

&JOB;ES/R073/S3;SS3A;

&ALGOL;
LIBRARY
ALGOL

&LINES;2000;

&LIST;

```
PROGRAM SS3A  SIMULATION OF NATIONAL ECONOMIC SYSTEM WITH VINTAGE
              MODEL AND POPULATION GROWTH                  M J HOWARTH;

"BEGIN""INTEGER"SIM,NSIM,LIFE,DIM,I,J,ST,Q,MAXQ,M,TEST;
       "BOOLEAN"DOUBLE;
       "REAL"ALPHA,BETA,GAMMA,RHO,DELTA,ETA,XI,ZETA,SIGMA,SW,SP;
       "REAL"V0,BIGT0,W0,K0,N0,U,V,A,BIGT,K,W,N,SF,H,RT;
       "REAL"THETA,R,C,LAMBDA,HST,C0,R0;
       "ARRAY"HISTV,HISTC[0:500];

"PROCEDURE"DERIVF(RT,Y,DY);
          "VALUE"RT;
          "REAL"RT;     "ARRAY"Y,DY;

"COMMENT"THIS PROCEDURE CALCULATES THE DERVATIVE OF THE STATE VECTOR
         DY IS THE OUTPUT VARIABLE AND RT REPRESENTS REAL TIME;
      "BEGIN""INTEGER"I,J;       "REAL"AGE,P,Q,NAGE,X;  "ARRAY"FUN[-1

"REAL""PROCEDURE"INTERP(FUN,X);
               "REAL"X;           "ARRAY"FUN;

"COMMENT" THIS PROCEDURE INTERPOLATES FOR FUN[X] IN THE ARRAY FUN[-1
          USING AITKEN'S ALGORITHM FOR ITERATED LAGRANGIAN INTERPOLAT

      "BEGIN""INTEGER"I,J;  "ARRAY"XX[-1:2],Y[-1:2,0:3];
            "FOR"I:=-1"STEP"1"UNTIL"2"DO"
            "BEGIN"XX[I]:=X-I;
                   Y[I,0]:=FUN[I];
                   "FOR"J:=1"STEP"1"UNTIL"I+1"DO"
                   Y[I,J]:=(XX[J-2]*Y[I,J-1]-XX[I]*Y[J-2,J-1])/(I-J
            "END"OF LOOP I;
            INTERP:=Y[2,3];
      "END"OF PROCEDURE INTERP;

"REAL""PROCEDURE"INTEGRAL(FUN);
               "ARRAY"FUN;

"COMMENT" THIS PROCEDURE EVALUATES THE INTEGRAL OF FUN[0:500] BETWEE
          (T-BIGT) AND T USING THE TRAPEZIUM RULE - GLOBAL VALUES OF
          P, Q AND SF ARE REQUIRED;
```

```
        "BEGIN""INTEGER"J;     "REAL"SUM;
             SUM:=(FUN[1]+FUN[I+1])/2;
             "FOR"J:=2"STEP"1"UNTIL"I"DO"SUM:=SUM+FUN[J];
             SUM:=SUM-(FUN[1]+FUN[0])*P/2;
             SUM:=SUM-Q/2*(FUN[I]*Q+FUN[I+1]*(2-Q));
             INTEGRAL:=SUM/SF;
        "END"OF PROCEDURE INTEGRAL;

   "COMMENT"      MODEL 3 - BASIC VINTAGE MODEL IN WHICH:
                                                  Y[1] IS N
                                                  Y[2] IS V AND
                                                  Y[3] IS BIGT;
            RST:=RT*SF;
            AGE:=1+SF*Y[3]+ENTIER(RST-0,1+6)-RST;
            I:=ENTIER(AGE);
            P:=RST-ENTIER(RST-0,1*6);
            Q:=I+1-AGE;
            HISTN[0]:=Y[1];
            V:=EXP(-BETA*RT)*INTEGRAL(HISTN);
            "FOR"J:=-1,0,1,2"DO"FUN[J]:=HISTN[I+1-J];
            NAGE:=INTERP(FUN,Q);
            NAGE:=INTERP(FUN,Q)/EXP(BETA*(RT-Y[3]));
            X:=-GAMMA+RHO/(1-V);
            DY[1]:=(Y[1]*(SP-ALPHA*SIGMA)-EXP(-ALPHA*Y[3])*((SP-SW)
            *(Y[1]+X*V)+X*SW*NAGE/ALPHA/EXP(BETA*Y[3]))))/SIGMA
            -Y[1]*BETA;
            X:=X/ALPHA;
            DY[2]:=Y[1]-BETA*V-NAGE*X*EXP(-BETA*Y[3])*V-Y[2];
            DY[3]:=1-X;
        "END"OF PROCEDURE DERIVF;

   "PROCEDURE"RKUTTA(N,H,X0,Y0,Y1);
            "VALUE"X0,Y0,H;
            "INTEGER"N;      "REAL"X0,H;     "ARRAY"Y0,Y1;

   "COMMENT" THIS PROCEDURE INTEGRATES THE N DIMENSIONAL SYSTEM OF EQUAT
            DY/DX=F(X,Y)OVER ONE STEP OF LENGTH H FROM (X0,Y0) USING A
            ORDER RUNGE-KUTTA ALGORITHM.  OUTPUT IS VIA THE ARRAY Y1 -
            PROCEDURE DERIVF MUST BE DECLARED GLOBALLY;

        "BEGIN""INTEGER"I,J;
             "ARRAY"Y,DY[1:N],K[1:N,1:3];
            DERIVF(X0,Y0,DY);
            "FOR"I:=1"STEP"1"UNTIL"N"DO"
            "BEGIN"K[I,1]:=DY[I];
                   Y[I]:=Y0[I]+H/2*DY[I];
            "END"OF LOOP I;
            DERIVF(X0+H/2,Y,DY);
            "FOR"I:=1"STEP"1"UNTIL"N"DO"
            "BEGIN"K[I,2]:=DY[I];
                   Y[I]:=Y0[I]+H/2*DY[I];
            "END"OF LOOP I;
            DERIVF(X0+H/2,Y,DY);
            "FOR"I:=1"STEP"1"UNTIL"N"DO"
            "BEGIN"K[I,3]:=DY[I];
                   Y[I]:=Y0[I]+H*DY[I];
            "END"OF LOOP I;
            DERIVF(X0+H,Y,DY);
            "FOR"I:=1"STEP"1"UNTIL"N"DO"Y1[I]:=Y0[I]+H/6*(K[I,1]+2*
```

```
                2]+2*K[I,3]+DY[I]);
        "END"OF PROCEDURE RKUTTA;

"PROCEDURE"HAMMING(N,H,X0,Y0,DY0,Y1,DY1,TOL,TEST);
            "VALUE"H,X0;
             "INTEGER"N,TOL,TEST;  "REAL"H,X0;  "ARRAY"Y0,DY0,Y1,DY1;

"COMMENT" THIS PROCEDURE PERFORMS THE SAME FUNCTION AS RKUTTA BUT US
          THE 4TH ORDER MILNE-HAMMING PREDICTOR-CORRECTOR METHOD.  I
          THE OUTPUT PARAMETER TEST IS 0 THEN THE RESULTS CONTAIN AN
          ERROR > 0,1*TOL, IF TEST >0 THE RESULTS ARE ACCEPTABLE AND
          TEST =2 THE STEP LENGTH MAY BE DOUBLED;

        "BEGIN""INTEGER"I,J;         "ARRAY"PRED,CORR,L,K[1:N];
              "SWITCH"SS:=S1,S2;
              "FOR"I:=1"STEP"1"UNTIL"N"DO"CORR[I]:=PRED[I]:=Y0[I,-3]+
              3*(2*DY0[I,0]-DY0[I,-1]+2*DY0[I,-2]);
    S1;       DERIVE(X0+H,CORR,DY1);
              "FOR"I:=1"STEP"1"UNTIL"N"DO"Y1[I]:=(9*Y0[I,0]-Y0[I,-2]+
              (DY1[I]+2*DY0[I,0]-DY0[I,-1]))/8;
              "FOR"I:=1"STEP"1"UNTIL"N"DO""IF"ABS((Y1[I]-CORR[I])/COR
              )>0,1*TOL/2"AND"ABS(Y1[I]-CORR[I])>0,1*TOL/20"THEN"
              "BEGIN""FOR"I:=1"STEP"1"UNTIL"N"DO"CORR[I]:=Y1[I];
                    "GOTO"S1;
              "END"OF CONDITION THEN;
              TEST:=2;
              "FOR"I:=1"STEP"1"UNTIL"N"DO"
              "BEGIN"L[I]:=ABS(Y1[I]-PRED[I]);
                    K[I]:=ABS(L[I]/PRED[I]);
                    "IF"K[I]>5*0,1*TOL"AND"L[I]>0,1*TOL/2"THEN"
                    "BEGIN"TEST:=0;
                          "GOTO"S2;
                    "END"OF CONDITION THEN;
                    "IF"K[I]>0,1*TOL/20"AND"L[I]>0,1*TOL/200"THEN"
                    TEST:=1;
              "END"OF LOOP I;
    S2;"END"OF PROCEDURE HAMMING;

"PROCEDURE"HISTORY;

"COMMENT" THIS PROCEDURE UPDATES THE HISTORICAL TIME PATH OF VARIABL
          SUBJECT TO A TIME DELAY.  THE HISTORICAL RECORD IS STORED
          INTERVALS OF ONE SIMULATION TIME UNIT AND HIST[0] HOLDS TH
          CURRENT VALUE OF THE VARIABLES;

        "BEGIN""INTEGER"I;
              "FOR"I:=500"STEP"-1"UNTIL"1"DO"
              "BEGIN"HISTN[I]:=HISTN[I-1];
                    HISTC[I]:=HISTC[I-1];
              "END"OF LOOP I;
        "END"OF PROCEDURE HISTORY;

"COMMENT" INPUT SYSTEM PARAMETERS AND EQUILIBRIUM VALUES FOLLOWED BY
          DETERMINANTS OF THE CURRENT SIMULATION RUNS.  THE SCALE FA
          SF, IS THE NUMBER OF SIMULATION TIME UNITS PER YEAR AND
          CORRESPONDS TO THE FINAL STORAGE AND OUTPUT INTERVAL;

        "READ"ALPHA,BETA,GAMMA,RHO,DELTA,XI,ETA,ZETA,A,SW,SP,SIGMA;
        "PRINT"''L2'PARAMETER VALUES OF SYSTEM ARE'L'',
        ''L'ALPHA=',ALIGNED(1,5),SAMELINE,ALPHA,
```

```
      ''L'BETA=',BETA,
      ''L'GAMMA=',GAMMA,
      ''L'RHO=',RHO,
      ''L'SIGMA=',SIGMA,
      ''L'DELTA=',DELTA,
      ''L'XI=',XI,
      ''L'ETA=',ETA,
      ''L'ZETA=',ZETA,
      ''L'COBB-DOUGLAS INDEX=',A,
      ''L'SAVINGS RATES,SW=',SW,'AND SP=',SP,''L'');
      "READ"V0,BIGT0,N0,C0,R0;
      "READ"NSIM;
      "FOR"SIM:=1"STEP"1"UNTIL"NSIM"DO"
      "BEGIN""READ"U,V,LIFE,SF;
             "FOR"I:=0"STEP"1"UNTIL"500"DO"HISTN[I]:=N0*EXP(-BETA*I/
             SF);
             RT:=0;
             C:=C0;
             "PRINT"''L5'RUN',SAMELINE,DIGITS(1),SIM,''L2'U=',
             ALIGNED(1,3),U,' AND V=',V,' INITIALLY'L2'SIMULATE',
             DIGITS(2),LIFE,' YEARS WITH OUTPUT INTERVAL OF',
             ALIGNED(2,0),12/SF,' MONTHS';
             "PRINT"''L10'TIME IN'S6'SHAPE OF'S6'RELATIVE'S7'PLANT'S7
             'EMPLOYMENT'S5'INTEGRATION'L' YEARS'S8'WAGES'S7'EMPLOYME
             ,''S6'LIFE'S7'ON NEW PLANT    SUBDIVISION'';

"COMMENT" ENTER INITIAL SET-UP PROCESS FOR PREDICTOR-CORRECTOR METHOD

"COMMENT" SET UP MODEL 3 - BASIC VINTAGE MODEL;

             DIM:=3;
             "BEGIN""ARRAY"Y,DY[1:3,-3:0],Y0,Y1,DY0,DY1,VAL,DVAL,OLD4
                    OLD5[1:3];      "SWITCH"SS:=L1,L2;
                    "REAL"IA,IB,IC,OLDT;

"PROCEDURE"UPDATE;

"COMMENT" THIS PROCEDURE UPDATES THE IMMEDIATE PAST RECORD OF THE STA
          VARIABLE TO PERMIT A FURTHER APPLICATION OF MHAMMING;  .

      "BEGIN""INTEGER"I,J;
             "FOR"I:=1"STEP"1"UNTIL"DIM"DO"
             "BEGIN"OLD5[I]:=OLD4[I];
                    OLD4[I]:=Y[I,-3];
                    "FOR"J:=-3,-2,-1"DO"
                    "BEGIN"Y[I,J]:=Y[I,J+1];
                           DY[I,J]:=DY[I,J+1];
                    "END"OF LOOP J;
                    Y[I,0]:=Y1[I];
                    DY[I,0]:=DY1[I];
             "END"OF LOOP I;
      "END"OF PROCEDURE UPDATE;

                    G:=64;
                    IA:=(2*SF*V+N0)*SIGMA;
                    IB:=2*SIGMA*N0*SF;
                    IC:=V*(SP*(1/U-1)+SW);
                    IA:=BIGT:=IA/IB;
                    IC:=IC/IB;
   L2:              OLDT:=BIGT;
```

```
                        BIGT:=IA-IC/EXP(ALPHA*OLDT);
                        BIGT:=-LN(1-BETA*BIGT)/BETA;
                        "IF"ABS(OLDT-BIGT)>0.1+4"THEN""GOTO"L2;
                        N:=2*SF*(V-(1-EXP(-BETA*BIGT))/BETA*N0)+N0;
                        Y[1,-3]:=Y0[1]:=N;
                        Y[2,-3]:=Y0[2]:=V;
                        Y[3,-3]:=Y0[3]:=BIGT;
                        HISTN[0]:=N;
                        HISTORY;

"COMMENT" END OF PARTICULAR MODEL;

                        H:=1/Q/SF;
                        "PRINT"''L'',ALIGNED(3,2),RT,SAMELINE,ALIGNED(6,5
                        U,ALIGNED(7,5),V,ALIGNED(6,4),BIGT,ALIGNED(7,5),N
                        ''S4'',DIGITS(3),Q;
                        "FOR"J:=-2,-1,0"DO"
                        "BEGIN"RKUTTA(DIM,H,(2+J)*H,Y0,VAL);
                               DERIVF((3+J)*H,VAL,DVAL);
                               "FOR"I:=1"STEP"1"UNTIL"DIM"DO"
                               "BEGIN"Y[I,J]:=Y0[I]:=VAL[I];
                                      DY[I,J]:=DVAL[I];
                               "END"OF LOOP I;
CARE
                        "END"OF LOOP J AND INITIAL SET-UP PROCESS;

"COMMENT" ENTER MAIN SIMULATION;

                        OLD4[3]:=OLD5[3]:=0;
                        MAXQ:=Q;
                        "FOR"ST:=1"STEP"1"UNTIL"LIFE*SF+0.1"DO"
                        "BEGIN""IF"ST=1"THEN"M:=Q-3"ELSE"M:=0;
                               RT:=ST/SF-M*H;
   L1:                         MHAMMING(DIM,H,RT,Y,DY,Y1,DY1,4,TEST);

"COMMENT" TEST TRUNCATION ERROR AND IF UNACCEPTABLE REDUCE STEP LENGT
          OF INTEGRATION;

                               "IF"TEST=0"THEN"
                               "BEGIN"H:=H/2;
                                      M:=M*2;
                                      Q:=Q*2;
                                      "IF"Q>MAXQ"THEN"MAXQ:=Q;

"COMMENT" RE-ENTER A SET-UP PROCESS FOR PREDICTOR-CORRECTOR METHOD
          HAVING HALVED THE STEP LENGTH OF INTEGRATION;

                                      "FOR"J:=1"STEP"1"UNTIL"DIM"DO"
                                      "BEGIN"OLD4[J]:=Y[J,-2];
                                             OLD5[J]:=0;
                                             VAL[J]:=OLD4[J];
                                      "END"OF LOOP J;
                                      "FOR"I:=-3,-2,-1,0"DO"
                                      "BEGIN"RKUTTA(DIM,H,RT+I*H-H,VAL,Y1
                                             DERIVF(RT+I*H,Y1,DY1);
                                             "FOR"J:=1"STEP"1"UNTIL"DIM"D
                                             "BEGIN"Y[J,I]:=VAL[J]:=Y1[J]
                                                    DY[J,I]:=DY1[J];
                                             "END"OF LOOP J;
                                      "END"OF LOOP I;
```

```
                                   "GOTO"L1;
                              "END"OF TEST FAIL;
                              M:=M-1;
                              RT:=RT+H;

"COMMENT" DOUBLE INTEGRATION STEP LENGTH IF APPROPRIATE;

                              DOUBLE:="FALSE";
                              "IF"TEST=2"AND"OLD5[3]"NE"0"AND"M=2*(M"DIV
                              "AND"Q>1
                              "THEN"
                              "BEGIN"H:=H*2;
                                   M:=M"DIV"2;
                                   Q:=Q"DIV"2;
                                   DOUBLE:="TRUE";
                              "END"OFTEST DOUBLE;

"COMMENT" TEST FOR OUTPUT POINT;

"COMMENT" OUTPUT ROUTINE FOR MODEL 4 - BASIC VINTAGE MODEL WITH BETA=

                              "IF"M=0"THEN"
                              "BEGIN"HISTN[0]:=N:=Y1[1];
                                   HISTN[0]:=N*EXP(BETA*RT);
                                   V:=Y1[2];
                                   BIGT:=Y1[3];
                                   U:=SP/(SIGMA*N*EXP(ALPHA*BIGT
                                   )/V-SW+SP);
                                   "PRINT"ALIGNED(3,2),RT,SAMELINE,
                                   ALIGNED(6,5),U,ALIGNED(7,5),V,ALIGN
                                   (8,4),BIGT,ALIGNED(7,5),N,''S4'',
                                   DIGITS(3),MAXQ;

"COMMENT" END OF PARTICULAR MODEL;

                                   HISTORY;
                                   MAXQ:=0;
                              "END"OF OUTPUT;

"COMMENT" ADVANCE SOLUTION PRIOR TO RE-ENTERING PREDICTOR-CORRECTOR
          INTEGRATION ROUTINE;

                              "IF""NOT"DOUBLE"THEN"UPDATE"ELSE"
                              "FOR"J:=1"STEP"1"UNTIL"DIM"DO"
                              "BEGIN"Y[J,0]:=Y1[J];
                                   Y[J,-2]:=Y[J,-3];
                                   Y[J,-3]:=OLD5[J];
                                   OLD4[J]:=OLD5[J]:=0;
                                   DY[J,0]:=DY1[J];
                                   DY[J,-2]:=DY[J,-3];
CARE
                              "END"OF DOUBLE=TRUE;
                              "IF"M"NE"0"THEN""GOTO"L1;
                         "END"OF LOOP ST AND CURRENT RUN;
                    "END"OF ARRAY DECLARATION BLOCK;
               "END"OF LOOP SIM;
"END"OF LOT;
   740   MC
  2526   CODE
  3266   TOTAL
```

RESULTS

&RUN
PROGRAMS
DRO

PARAMETER VALUES OF SYSTEM ARE

ALPHA= 0,03000
BETA=-0,02000
GAMMA= 0,02000
RHO= 0,00100
SIGMA= 3,60000
DELTA= 0,00000
XI= 0,00000
ETA= 0,00000
ZETA= 0,00000
COBB-DOUGLAS INDEX= 0,00000
SAVINGS RATES,SW= 0,08000AND SP= 0,55000

RUN 1

U= 0,780 AND V= 0,985 INITIALLY

SIMULATE 2 YEARS WITH OUTPUT INTERVAL OF 2 MONTHS

TIME IN YEARS	SHARE OF WAGES	RELATIVE EMPLOYMENT	PLANT LIFE	EMPLOYMENT ON NEW PLANT.	INTEGRAT SUBDIVIS
0,00	0,78000	0,98500	19,9840	0,03532	64
0,17	0,78082	0,98136	19,9368	0,03513	64
0,33	0,78115	0,98000	19,9281	0,03505	4
0,50	0,78130	0,97940	19,9325	0,03500	4
0,67	0,78139	0,97912	19,9421	0,03497	4
0,83	0,78143	0,97900	19,9546	0,03495	4
1,00	0,78147	0,97893	19,9680	0,03493	4
1,17	0,78149	0,97889	19,9822	0,03491	4
1,33	0,78150	0,97887	19,9966	0,03489	4
1,50	0,78151	0,97886	20,0114	0,03487	4
1,67	0,78152	0,97884	20,0262	0,03485	4
1,83	0,78152	0,97883	20,0410	0,03484	4
2,00	0,78152	0,97881	20,0559	0,03482	4

A2.4 Program ES4

´TITLE` Equilibrium solution for basic putty-clay model.

´FUNCTION` To determine the coordinates of the unique equilibrium of the system in question. The non-linear system of algebraic equations is solved using a two-dimensional Newton-Raphson algorithm.

´USAGE` The user is required to supply the following item of data in a sequence as specified in the line of the program quoted:

(i) system parameters (line 5).

´EXAMPLE DATA`

´Item (i)` 0·03 0 0·02 0·001 3·6 0·08 0·55 0·25

```
** ICL 4130 DES2 SYSTEM  :  SLAVE BIG        :  CORE 70K  :  VOL  4


&JOB:ES/R073/ES:EQUILIBRIUM SOLUTION OF NATIONAL ECONOMIC SYSTEMS  MJH;


 &ALGOL;
LIBRARY
ALGOL


 &LIST;

PROGRAM ES4   EQUILIBRIUM SOLUTION FOR PUTTY CLAY MODEL   M J HOWARTH

"BEGIN""REAL"ALPHA,BETA,GAMMA,RHO,A,SIGMA,SW,SP,R,C,BIGT,W,V,U,N;
        "REAL"EXPA,EXPR,F,G,FT,FR,GT,GR,DEL,NBIGT,NR;   "SWITCH"SS:=L1
        "READ"ALPHA,BETA,GAMMA,RHO,SIGMA,SW,SP,A;
        "PRINT"''L5'PARAMETER VALUES OF SYSTEM ARE'L2'',
        'ALPHA=',ALIGNED(1,5),SAMELINE,ALPHA,
        ''L'BETA=',BETA,
        ''L'GAMMA=',GAMMA,
        ''L'RHO=',RHO,
        ''L'SIGMA=',SIGMA,
        ''L'COBB-DOUGLAS INDEX=',A,
        ''L'SAVINGS RATES,SW=',SW,' AND SP=',SP;
        R:=ALPHA;
        BIGT:=20;
    L1; "IF"ABS(R-ALPHA)<0,0001 "THEN"R:=ALPHA+0,0001;
        EXPA:=EXP(-ALPHA*BIGT);
        EXPR:=EXP(-R*BIGT);

"COMMENT"  SOLUTION BY NEWTON-RAPHSON ALGORITHM - COMPUTE FUNCTION AN
           PARTIAL DERIVATES FOR CURRENT APPROXIMATION;

        F:=SP*R/ALPHA*(1- EXPA)-R*BIGT*EXPA*(SP-SW)-A*(1-EXPR);
        G:=(EXPA-EXPR)/(R-ALPHA)-(1-A)/R*(1-EXPR);
        FR:=SP/ALPHA*(1-EXPA)-BIGT*(EXPA*(SP-SW)+A*EXPR);
        FT:=SP*R*EXPA-R*(EXPA*(SP-SW)*(1-ALPHA*BIGT)+A*EXPR);
        GR:=-(EXPA-EXPR)/(R-ALPHA)^2+BIGT*EXPR/(R-ALPHA)+
        (1-A)*((1-EXPR)/R^2-BIGT/R*EXPR);
        GT:=(R*EXPR-ALPHA*EXPA)/(R-ALPHA)-(1-A)*EXPR;
        DEL:=FR*GT-GR*FT;
        NR:=R-(F*GT-G*FT)/DEL;
        NBIGT:=BIGT-(G*FR-F*GR)/DEL;
        "IF"ABS((NBIGT-BIGT)/BIGT)>0,1^5"OR"ABS((NR-R)/R)>0.1^5"THEN"
        "BEGIN"BIGT:=NBIGT;
               R:=NR;
               "GOTO"L1;
        "END"OF RELATION TRUE;
        V:=1-RHO/(ALPHA+GAMMA);
        C:=(A/SIGMA/R*(1-EXPR))^(1/(1-A));
        W:=C^A/SIGMA*EXPA;
        N:=V/BIGT;
        U:=SP/(SP-SW+SIGMA*C^(1-A)/BIGT/EXPA);
        "PRINT"''L5'EQUILIRIUM SOLUTION IS'L2'',
        'SHARE OF WAGES=',SAMELINE,FREEPOINT(5),U,''L'',
```

```
              'RELATIVE EMPLOYMENT=',V,
              ''L'LIFE OF PLANT=',BIGT,
              ''L'PROPORTION OF LABOUR EMPLOYED ON NEW PLANT=',H,
              ''L'PER CAPITA INVESTMENT CONSTANT=',C,
              ''L'RATE OF RETURN ON INVESTMENTS=',R,
              ''L'WAGE CONSTANT=',W;
     "END"OF LOT;
 332    MC
 588  CODE
 920 TOTAL


 &RUN;
PROGRAME
DRO
```

RESULTS

```
PARAMETER VALUES OF SYSTEM ARE

ALPHA= 0,03000
BETA= 0,00000
GAMMA= 0,02000
RHO= 0,00100
SIGMA= 3,60000
COBB-DOUGLAS INDEX= 0,25000
SAVINGS RATES,SW= 0,08000 AND SP= 0,55000



EQUILIRIUM SOLUTION IS

SHARE OF WAGES= ,77060
RELATIVE EMPLOYMENT= ,98000
LIFE OF PLANT= 16,678
PROPORTION OF LABOUR EMPLOYED ON NEW PLANT= ,05876
PER CAPITA INVESTMENT CONSTANT= ,60342
RATE OF RETURN ON INVESTMENTS= ,06973
WAGE CONSTANT= ,14844



STORE LEFT 61386  USED  5227


 &END;
CPU TIME =  0000  08,638  REAL TIME   00 00 24
   A
```

A2.5 Program SS4

'TITLE' Simulation of national economic system with basic putty-clay model

'FUNCTION' As for program SS3 but catering for non-constant capital-output ratio. Detailed discussion may likewise be found in ch. 6.

'USAGE' The user is required to supply the following items of data in a sequence as specified in the line of the program quoted:

(i) system parameters (line 263)

(ii) coordinates of system equilibrium (line 276)

(iii) number of runs

(iv) initial conditions, run length in years (LIFE) and scale factor (SF) for each run (line 279)

'EXAMPLE DATA'

'Item (i)' 0·033 0 0·0435 0·00095 0 0
0 0 0·25 0 1 3·53

'Item (ii)' 0·98759 16·673 0·05923 0·87946 0·033

'Item (iii)' 1

'Item (iv)' 0·780 0·988 1 24

```
** ICL 4130 DES2 SYSTEM  :  SLAVE BIG       :  CORE 70K  :  VOL  4

&JOB;ES/R073/SS;SIMULATION OF NATIONAL ECONOMIC SYSTEMS  H,J, HOWARTH;

&TIME;10;

&ALGOL;
LIBRARY
ALGOL

&LIST;

&LINES;1000;

PROGRAM SS4  SIMULATION OF NATIONAL ECONOMIC SYSTEM WITH BASIC
             PUTTY CLAY MODEL                          M J HOWARTH;

"BEGIN""INTEGER"SIM,NSIM,LIFE,DIM,I,J,ST,Q,MAXQ,M,TEST;
       "BOOLEAN"DOUBLE,START;
       "REAL"ALPHA,BETA,GAMMA,RHO,DELTA,ETA,XI,ZETA,SIGMA,SW,SP;
       "REAL"V0,BIGT0,W0,K0,N0,U,V,A,BIGT,K,W,N,SF,H,RT;
       "REAL"THETA,R,C,LAMBDA,RST,C0,R0,CAGE;
       "ARRAY"HISTN,HISTC[0:500];

"PROCEDURE"DERIVE(RT,Y,DY);
          "VALUE"RT;
          "REAL"RT;    "ARRAY"Y,DY;

"COMMENT" THIS PROCEDURE CALCULATES THE DERIVATIVE OF THE STATE VECT
         Y,  DY IS THE OUTPUT VARIABLE AND RT REPRESENTS REAL TIME;

      "BEGIN""INTEGER"I,J;
             "REAL"AGE,P,Q,NAGE,DCAGE,L,K,F0,G0,RU,RD,CU,CD,EXP0,X;
              "ARRAY"FUN,GUN[-1:2];
             "SWITCH"SS:=Z1,Z2,Z3,Z4,Z5,Z6;
             "BOOLEAN"CDOWN,CUP;


"REAL""PROCEDURE"INTERP(FUN,X);
                 "REAL"X;           "ARRAY"FUN;

"COMMENT" THIS PROCEDURE INTERPOLATES FOR FUN[X] IN THE ARRAY FUN[-:
         USING AITKEN'S ALGORITM FOR ITERATED LAGRANGIAN INTERPOLA'

      "BEGIN""INTEGER"I,J;  "ARRAY"XX[-1:2],Y[-1:2,0:3];
             "FOR"I:=-1"STEP"1"UNTIL"2"DO"
             "BEGIN"XX[I]:=X-I;
                    Y[I,0]:=FUN[I];
                    "FOR"J:=1"STEP"1"UNTIL"I+1"DO"
                    Y[I,J]:=(XX[J-2]*Y[I,J-1]-XX[I]*Y[J-2,J-1])/(I-
```

```
            "END"OF LOOP I;
            INTERP:=Y[2,3];
      "END"OF PROCEDURE INTERP;

"REAL""PROCEDURE"INTEGRAL(FUN);
                "ARRAY"FUN;

"COMMENT" THIS PROCEDURE EVALUATES THE INTEGRAL OF FUN[0:500] BETWEEN
          (T-BIGT) AND T USING THE TRAPEZIUM RULE - GLOBAL VALUES OF
          P, Q AND SF ARE REQUIRED;

      "BEGIN""INTEGER"J;    "REAL"SUM;
            SUM:=(FUN[1]+FUN[I+1])/2;
            "FOR"J:=2"STEP"1"UNTIL"I"DO"SUM:=SUM+FUN[J];
            SUM:=SUM+(FUN[1]+FUN[0])*P/2;
            SUM:=SUM-Q/2*(FUN[I]*Q+FUN[I+1]*(2-Q));
            INTEGRAL:=SUM/SF;
      "END"OF PROCEDURE INTEGRAL;

"COMMENT" MODEL 4     THE BASIC PUTTY-CLAY MODEL IN WHICH:
                            Y[1] IS C*N
                            Y[2] IS V
                            Y[3] IS BIGT;

            "REAL""PROCEDURE"F(C,R);
                  "REAL"C,R;
                  "BEGIN""REAL"X;
                        THETA:=(ALPHA*Y[3]+A*(LN(C)-LN(CAGE)))/LAMB
                        EXPO:=EXP(-R*THETA);
                        X:="IF"ABS(R-LAMBDA)<0.1+6"THEN"THETA*EXPO
                        "ELSE"(EXP(-LAMBDA*THETA)-EXPO)/(R-LAMBDA);
                         "IF"ABS(R)<0.1+6"THEN"
                         F:=SIGMA*C+C+A*(X-THETA)"ELSE"
                        F:=SIGMA*C+C+A*(X-(1-EXPO)/R);
                  "END"OF PROCEDURE F;

            "REAL""PROCEDURE"G(C,R);
                  "REAL"C,R;
                  "BEGIN"THETA:=(ALPHA*Y[3]+A*(LN(C)-LN(CAGE)))/LAMB
                         "IF"ABS(R)<0.1+6"THEN"
                         G:=SIGMA*C/A-C+A*THETA"ELSE"
                        G:=SIGMA*C/A-C+A/R*(1-EXP(-R*THETA));
                  "END"OF PROCEDURE G;

            "PROCEDURE"SEARCHR;
                    "BEGIN""BOOLEAN"RUP,RDOWN;
                          "SWITCH"SS:=L1,L2,L3,L4,L5;
                          K:=0.001;
                          RUP:=RDOWN:="FALSE";
                  L3:     F0:=F(C,R);
                          RU:=RD:=R;
                  L5:     "IF"RDOWN"THEN""GOTO"L4;
                  L2:     R:=RU+K;
                          "IF"F0*F(C,R)"LE"0"THEN"
                          "BEGIN"R:=RU;
                                 RUP:="TRUE";
                                 "GOTO"L1;
                          "END"OF RELATION TRUE;
                          RU:=R;
                  L4:     "IF"RUP"THEN""GOTO"L2;
```

```
                                R:=RD-K;
                                "IF"F0*F(C,R)"LE"0"THEN"
                                "BEGIN"R:=RD;
                                        RDOWN:="TRUE";
                                        "GOTO"L1;
                                "END"OF RELATION TRUE;
                                RD:=R;
                                "GOTO"L5;
                        L1:      "IF"K>5*0,1+6"THEN"
                                "BEGIN"K:=K/10;
                                        "GOTO"L3;
                                "END"OF RELATION TRUE;
                                "IF"RDOWN"THEN"R:=R-K;
                         "END"OF PROCEDURE SEARCHR;

               RST:=RT*SF;
               J:="IF"M=0"OR"M=1"THEN"ENTIER(RST-0,1+6)
               "ELSE"ENTIER(RST+0,1+6);
               AGE:=1+SF*Y[3]+J-RST;
               I:=ENTIER(AGE);
               P:=RST-J;
               Q:=I+1-AGE;
               "FOR"J:=-1,0,1,2"DO"
               "BEGIN"FUN[J]:=HISTN[I+1-J];
                       GUN[J]:=HISTC[I+1-J];
               "END"OF LOOP J;
               NAGE:=INTERP(FUN,Q);
               CAGE:=INTERP(GUN,Q);
               DCAGE:=(HISTC[I]-HISTC[I+1])*SF;
               LAMBDA:=-GAMMA+RHO/(1-Y[2]);

"COMMENT" SOLVE THE ALGEBRAIC EQUATIONS FOR C AND R BY TWO-DIMENSION
          ROOT SEARCH;

               L:=0,1;
               CUP:=CDOWN:="FALSE";
     Z4:       SEARCHR;
               G0:=G(C,R);
               CU:=CD:=C;
     Z5:       "IF"CDOWN"THEN""GOTO"Z1;
     Z2:       C:=CU+L;
               SEARCHR;
               "IF"G0*G(C,R)"LE"0"THEN"
               "BEGIN"C:=CU;
                       CUP:="TRUE";
                       "GOTO"Z3;
               "END"OF RELATION TRUE;
               CU:=C;
               "IF"CUP"THEN""GOTO"Z2;
     Z1:       C:=CD-L;
               SEARCHR;
               "IF"G0*G(C,R)"LE"0"THEN"
               "BEGIN"C:=CD;
                       CDOWN:="TRUE";
                       "GOTO"Z3;
               "END"OF RELATION TRUE;
               CD:=C;
               "GOTO"Z5;
     Z3:       "IF"L>0,0005"THEN"
               "BEGIN"L:=L/10;
```

```
                    "GOTO"Z4;
                "END"OF RELATION TRUE;

"COMMENT" C AND R ARE NOW DETERMINED;

                "IF"START"THEN""GOTO"Z6;
                EXPO:=EXP(-ALPHA*Y[3]);
                N:=Y[1]/C;
                HISTN[0]:=N;
                V:=INTEGRAL(HISTN);
                LAMBDA:=-GAMMA+RHO/(1-V);
                X:=LAMBDA/(A*DCAGE/CAGE+ALPHA);
                DY[1]:=(SP-SW)*EXPO*(CAGE+A*(ALPHA*X*V+N-NAGE*X)+A*CAGE
                        (A-1)*V*DCAGE*X);
                DY[1]:=(SP*(C+A*N-CAGE+A*NAGE*EXPO*X)-DY[1])/SIGMA-ALPH
                        C*N;
                DY[2]:=N-NAGE*X;
                DY[3]:=1-X;
      Z6:"END"OF PROCEDURE DERIVF;

"PROCEDURE"RKUTTA(N,H,X0,Y0,Y1);
            "VALUE"X0,Y0,H;
            "INTEGER"N;      "REAL"X0,H;     "ARRAY"Y0,Y1;

"COMMENT" THIS PROCEDURE INTEGRATES THE N DIMENSIONAL SYSTEM OF EQUA
          DY/DX=F(X,Y)OVER ONE STEP OF LENGTH H FROM (X0,Y0) USING A
          ORDER RUNGE-KUTTA ALGORITHM;  OUTPUT IS VIA THE ARRAY Y1 -
          PROCEDURE DERIVF MUST BE DECLARED GLOBALLY;

        "BEGIN""INTEGER"I,J;
                "ARRAY"Y,DY[1:N],K[1:N,1:3];
                DERIVF(X0,Y0,DY);
                "FOR"I:=1"STEP"1"UNTIL"N"DO"
                "BEGIN"K[I,1]:=DY[I];
                        Y[I]:=Y0[I]+H/2*DY[I];
                "END"OF LOOP I;
                DERIVF(X0+H/2,Y,DY);
                "FOR"I:=1"STEP"1"UNTIL"N"DO"
                "BEGIN"K[I,2]:=DY[I];
                        Y[I]:=Y0[I]+H/2*DY[I];
                "END"OF LOOP I;
                DERIVF(X0+H/2,Y,DY);
                "FOR"I:=1"STEP"1"UNTIL"N"DO"
                "BEGIN"K[I,3]:=DY[I];
                        Y[I]:=Y0[I]+H*DY[I];
                "END"OF LOOP I;
                DERIVF(X0+H,Y,DY);
                "FOR"I:=1"STEP"1"UNTIL"N"DO"Y1[I]:=Y0[I]+H/6*(K[I,1]+2*
                2]+2*K[I,3]+DY[I]);
        "END"OF PROCEDURE RKUTTA;

"PROCEDURE"MHAMMING(N,H,X0,Y0,DY0,Y1,DY1,TOL,TEST);
            "VALUE"H,X0;
            "INTEGER"N,TOL,TEST;  "REAL"H,X0;  "ARRAY"Y0,DY0,Y1,DY1;

"COMMENT" THIS PROCEDURE PERFORMS THE SAME FUNCTION AS RKUTTA BUT US
          THE 4TH ORDER MILNE-HAMMING PREDICTOR-CORRECTOR METHOD.  I
          THE OUTPUT PARAMETER TEST IS 0 THEN THE RESULTS CONTAIN AN
          ERROR > 0.1*TOL, IF TEST >0 THE RESULTS ARE ACCEPTABLE AN
          TEST =2 THE STEP LENGTH MAY BE DOUBLED;
```

```
        "BEGIN""INTEGER"I,J;            "ARRAY"PRED,CORR,L,K[1:N];
              "SWITCH"SS:=S1,S2;
              "FOR"I:=1"STEP"1"UNTIL"N"DO"CORR[I]:=PRED[I]:=Y0[I,-3]+4
              3*(2*DY0[I,0]-DY0[I,-1]+2*DY0[I,-2]);
    S1;       DERIVF(X0+H,CORR,DY1);
              "FOR"I:=1"STEP"1"UNTIL"N"DO"Y1[I]:=(9*Y0[I,0]-Y0[I,-2]+3
              (DY1[I]+2*DY0[I,0]-DY0[I,-1]))/8;
              Y1[2]:=V;
              "FOR"I:=1"STEP"1"UNTIL"N"DO""IF"ABS((Y1[I]-CORR[I])/CORR
              )>0.1*TOL/2"AND"ABS(Y1[1]-CORR[I])>0.1*TOL/20"THEN"
              "BEGIN""FOR"I:=1"STEP"1"UNTIL"N"DO"CORR[I]:=Y1[I];
                    "GOTO"S1;
              "END"OF CONDITION THEN;
              TEST:=2;
              "FOR"I:=1"STEP"1"UNTIL"N"DO"
              "BEGIN"L[I]:=ABS(Y1[I]-PRED[I]);
                    K[I]:=ABS(L[I]/PRED[I]);
                    "IF"K[I]>5*0.1*TOL"AND"L[I]>0.1*TOL/2"THEN"
                    "BEGIN"TEST:=0;
                          "GOTO"S2;
                    "END"OF CONDITION THEN;
                    "IF"K[I]>0.1*TOL/20"AND"L[I]>0.1*TOL/200"THEN"
                    TEST:=1;
              "END"OF LOOP I;
    S2;"END"OF PROCEDURE HHAMMING;

"PROCEDURE"HISTORY;

"COMMENT" THIS PROCEDURE UPDATES THE HISTORICAL TIME PATH OF VARIABLE
          SUBJECT TO A TIME DELAY,  THE HISTORICAL RECORD IS STORED A
          INTERVALS OF ONE SIMULATION TIME UNIT AND HIST[0] HOLDS THE
          CURRENT VALUE OF THE VARIABLES;

        "BEGIN""INTEGER"I;
              "FOR"I:=500"STEP"-1"UNTIL"1"DO"
              "BEGIN"HISTN[I]:=HISTN[I-1];
                    HISTC[I]:=HISTC[I-1];
              "END"OF LOOP I;
        "END"OF PROCEDURE HISTORY;

"COMMENT" INPUT SYSTEM PARAMETERS AND EQUILIBRIUM VALUES FOLLOWED BY
          DETERMINANTS OF THE CURRENT SIMULATION RUNS,  THE SCALE FAC
          SF, IS THE NUMBER OF SIMULATION TIME UNITS PER YEAR AND
          CORRESPONDS TO THE FINAL STORAGE AND OUTPUT INTERVAL;

        "READ"ALPHA,BETA,GAMMA,RHO,DELTA,XI,ETA,ZETA,A,SW,SP,SIGMA;
        "PRINT"''L2'PARAMETER VALUES OF SYSTEM ARE'L'',
        ''L'ALPHA=',ALIGNED(1,5),SAMELINE,ALPHA,
        ''L'BETA=',BETA,
        ''L'GAMMA=',GAMMA,
        ''L'RHO=',RHO,
        ''L'SIGMA=',SIGMA,
        ''L'DELTA=',DELTA,
        ''L'XI=',XI,
        ''L'ETA=',ETA,
        ''L'ZETA=',ZETA,
        ''L'COBB-DOUGLAS INDEX=',A,
        ''L'SAVINGS RATES,SW=',SW,'AND SP=',SP,''L'';
        "READ"V0,BIGT0,N0,C0,R0;
```

```
        "READ"NSIM;
        "FOR"SIM:=1"STEP"1"UNTIL"NSIM"DO"
        "BEGIN""READ"U,V,LIFE,SF;
                "FOR"I:=0"STEP"1"UNTIL"500"DO"
                "BEGIN"HISTN[I]:=N0;
                       HISTC[I]:=C0;
                "END"OF LOOP I;
                RT:=0;
                C:=C0;
                BIGT:=BIGT0;
                R:=R0;
                "PRINT"''L5'RUN',SAMELINE,DIGITS(1),SIM,''L2'U=',
                ALIGNED(1,3),U,' AND V=',V,' INITIALLY'L2'SIMULATE',
                DIGITS(2),LIFE,' YEARS WITH OUTPUT INTERVAL OF',
                ALIGNED(2,2),12/SF,' MONTHS';
                "PRINT"''L10'TIME IN'S6'SHARE OF'S6'RELATIVE'S7'PLANT'S7
                'EMPLOYMENT'S5'INTEGRATION'L' YEARS'S8'WAGES'S7'EMPLOYME
                ,''S6'LIFE'S7'ON NEW PLANT    SUBDIVISION';
                "PRINT"''S5'C'S12'R';

"COMMENT" ENTER INITIAL SET-UP PROCESS FOR PREDICTOR-CORRECTOR METHOD

"COMMENT" SET UP MODEL 4 - BASIC PUTTY-CLAY MODEL;

                DIM:=3;
                "BEGIN""ARRAY"Y,DY[1:3,-3:0],Y0,Y1,DY0,DY1,VAL,DVAL,OLD4
                        OLD5[1:3];      "SWITCH"SS:=L1,L2;
                        "REAL"IA,IB,IC,OLDT,OLDC;

"PROCEDURE"UPDATE;

"COMMENT" THIS PROCEDURE UPDATES THE IMMEDIATE PAST RECORD OF THE STA
          VARIABLE TO PERMIT A FURTHER APPLICATION OF HHAMMING;

        "BEGIN""INTEGER"I,J;
                "FOR"I:=1"STEP"1"UNTIL"DIM"DO"
                "BEGIN"OLD5[I]:=OLD4[I];
                        OLD4[I]:=Y[I,-3];
                        "FOR"J:=-3,-2,-1"DO"
                        "BEGIN"Y[I,J]:=Y[I,J+1];
                               DY[I,J]:=DY[I,J+1];
                        "END"OF LOOP J;
                        Y[I,0]:=Y1[I];
                        DY[I,0]:=DY1[I];
                "END"OF LOOP I;
        "END"OF PROCEDURE UPDATE;

                        Q:=32;
                        START:="TRUE";
                        Y0[1]:=C0*N0;
                        Y0[2]:=V;
                        IA:=(2*SF*V+N0)*SIGMA;
                        IB:=2*SIGMA*N0*SF;
                        IC:=V*(SP*(1/U-1)+SW)*C0*A;
                        IA:=IA/IB;
    L2:                 OLDT:=BIGT;
                        OLDC:=C;
                        Y0[3]:=BIGT;
                        N:=IC/EXP(ALPHA*BIGT)/C/SIGMA;
                        Y0[1]:=C*N;
```

```
                    DERIVF(0,Y0,Y1);
                    BIGT:=IA-IC/C/IB/EXP(ALPHA*OLDT);
                    "IF"ABS(OLDT-BIGT)>0,1+3"OR"ABS(OLDC-C)>0,1+4
                    "THEN""GOTO"L2;
                    START:="FALSE";
                    Y[1,-3]:=Y0[1]:=C*N;
                    Y[2,-3]:=Y0[2]:=V;
                    Y[3,-3]:=Y0[3]:=BIGT;

"COMMENT" END OF PARTICULAR MODEL;

                    HISTN[0]:=N;
                    HISTC[0]:=C;
                    HISTORY;
                    H:=1/Q/SF;
                    "PRINT"''L'',ALIGNED(3,2),RT,SAMELINE,ALIGNED(6,5
                    U,ALIGNED(7,5),V,ALIGNED(8,4),BIGT,ALIGNED(7,5),N
                    ''S4'',DIGITS(3),Q;
                    "PRINT"SAMELINE,''S8'',ALIGNED(3,3),C,ALIGNED
                    (7,4),R;
                    "FOR"J:=-2,-1,0"DO"
                    "BEGIN"M:=0-J-2;
                          RKUTTA(DIM,H,(2+J)*H,Y0,VAL);
                          DERIVF((3+J)*H,VAL,DVAL);
                          "FOR"I:=1"STEP"1"UNTIL"DIM"DO"
                          "BEGIN"Y[I,J]:=Y0[I]:=VAL[I];
                                DY[I,J]:=DVAL[I];
                          "END"OF LOOP I;
CARE
                    "END"OF LOOP J AND INITIAL SET-UP PROCESS;

"COMMENT" ENTER MAIN SIMULATION;

                    OLD4[3]:=OLD5[3]:=0;
                    MAXQ:=Q;
                    "FOR"ST:=1"STEP"1"UNTIL"LIFE*SF+0,1"DO"
                    "BEGIN""IF"ST=1"THEN"M:=0-3"ELSE"M:=0;
                          RT:=ST/SF-M*H;
    L1:                  MHAMMING(DIM,H,RT,Y,DY,Y1,DY1,5,TEST);

"COMMENT" TEST TRUNCATION ERROR AND IF UNACCEPTABLE REDUCE STEP LENGT
          OF INTEGRATION;

                          "IF"TEST=0"THEN"
                          "BEGIN"H:=H/2;
                                M:=M*2;
                                Q:=Q*2;
                                "IF"Q>MAXQ"THEN"MAXQ:=Q;

"COMMENT" RE-ENTER A SET-UP PROCESS FOR PREDICTOR-CORRECTOR METHOD
          HAVING HALVED THE STEP LENGTH OF INTEGRATION;

                                "FOR"J:=1"STEP"1"UNTIL"DIM"DO"
                                "BEGIN"OLD4[J]:=Y[J,-2];
                                      OLD5[J]:=0;
                                      VAL[J]:=OLD4[J];
                                "END"OF LOOP J;
                                "FOR"I:=-3,-2,-1,0"DO"
                                "BEGIN"RKUTTA(DIM,H,RT+I*H-H,VAL,Y
                                      DERIVF(RT+I*H,Y1,DY1);
```

```
                                                "FOR"J:=1"STEP"1"UNTIL"DIM"D
                                                "BEGIN"Y[J,I]:=VAL[J]:=Y1[J]
                                                         DY[J,I]:=DY1[J];
                                                "END"OF LOOP J;
                                         "END"OF LOOP I;
                                         "GOTO"L1;
                                    "END"OF TEST FAIL;
                                    M:=M-1;
                                    RT:=RT+H;

"COMMENT" DOUBLE INTEGRATION STEP LENGTH IF APPROPRIATE;

                                    DOUBLE:="FALSE";
                                    "IF"TEST=2"AND"OLD5[3]"NE"0"AND"H=2*(H"DIV
                                    "THEN"
                                    "BEGIN"H:=H*2;
                                         M:=M"DIV"2;
                                         Q:=Q"DIV"2;
                                         DOUBLE:="TRUE";
                                    "END"OFTEST DOUBLE;

"COMMENT" TEST FOR OUTPUT POINT;

"COMMENT" OUTPUT ROUTINE FOR MODEL 4 - BASIC PUTTY-CLAY MODEL;

                                    "IF"M=0"THEN"
                                    "BEGIN"DERIVF(RT,Y1,VAL);
                                         HISTN[0]:=N:=Y1[1]/C;
                                         V:=Y1[2];
                                         BIGT:=Y1[3];
                                         HISTC[0]:=C;
                                         U:=SP/(SP-SW+SIGMA*C*N*EXP(ALPHA*BI
                                         /V/CAGE+A);
                                         "PRINT"ALIGNED(3,2),RT,SAMELINE,
                                         ALIGNED(6,5),U,ALIGNED(7,5),V,ALIGN
                                         (8,4),BIGT,ALIGNED(7,5),N,''S4'',
                                         DIGITS(3),MAXQ;
                                         "PRINT"SAMELINE,''S8'',ALIGNED(3,3)
                                         ALIGNED(7,4),R;

"COMMENT" END OF PARTICULAR MODEL;

                                         HISTORY;
                                         MAXQ:=0;
                                    "END"OF OUTPUT;

"COMMENT" ADVANCE SOLUTION PRIOR TO RE-ENTERING PREDICTOR-CORRECTOR
          INTEGRATION ROUTINE;

                                    "IF""NOT"DOUBLE"THEN"UPDATE"ELSE"
                                    "FOR"J:=1"STEP"1"UNTIL"DIM"DO"
                                    "BEGIN"Y[J,0]:=Y1[J];
                                         Y[J,-2]:=Y[J,-3];
                                         Y[J,-3]:=OLD5[J];
                                         OLD4[J]:=OLD5[J]:=0;
                                         DY[J,0]:=DY1[J];
                                         DY[J,-2]:=DY[J,-3];
CARE
                                    "END"OF DOUBLE=TRUE;
                                    "IF"M"NE"0"THEN""GOTO"L1;
```

```
                      "END"OF LOOP ST AND CURRENT RUN;
               "END"OF ARRAY DECLARATION BLOCK;
        "END"OF LOOP SIM;
  "END"OF LOT;
 892    MC
3188  CODE
4080 TOTAL
```

```
&RUN;
PROGRAMS
DRO
```

RESULTS

PARAMETER VALUES OF SYSTEM ARE .

ALPHA= 0,03300
BETA= 0,00000
GAMMA= 0,04350
RHO= 0,00095
SIGMA= 3,53000
DELTA= 0,00000
XI= 0,00000
ETA= 0,00000
ZETA= 0,00000
COBB-DOUGLAS INDEX= 0,25000
SAVINGS RATES,SW= 0,00000AND SP= 1,00000

RUN 1

U= 0,780 AND V= 0,988 INITIALLY

SIMULATE 1 YEARS WITH OUTPUT INTERVAL OF 0,50 MONTHS

TIME IN YEARS	SHAPE OF WAGES	RELATIVE EMPLOYMENT	PLANT LIFE	EMPLOYMENT ON NEW PLANT	INTEGRA SUBDIVI
0,00	0,78000	0,98800	16,6853	0,04629	32
0,04	0,78008	0,98750	16,6843	0,05112	32
0,08	0,78011	0,98730	16,6859	0,05313	16
0,12	0,78013	0,98722	16,6885	0,05398	8
0,17	0,78015	0,98716	16,6915	0,05434	8
0,21	0,78016	0,98717	16,6946	0,05450	4
0,25	0,78017	0,98717	16,6978	0,05459	2
0,29	0,78018	0,98716	16,7011	0,05462	2
0,33	0,78019	0,98717	16,7043	0,05464	2

0,37	0,78021	0,98717	16,7075	0,05467	2
0,42	0,78022	0,98717	16,7108	0,05467	2
0,46	0,78023	0,98717	16,7140	0,05471	2
0,50	0,78024	0,98717	16,7172	0,05472	2
0,54	0,78025	0,98717	16,7204	0,05474	2
0,58	0,78026	0,98717	16,7236	0,05475	2
0,63	0,78027	0,98717	16,7267	0,05477	2
0,67	0,78028	0,98718	16,7299	0,05478	2
0,71	0,78029	0,98718	16,7330	0,05480	2
0,75	0,78030	0,98718	16,7362	0,05481	2
0,79	0,78031	0,98718	16,7393	0,05484	2
0,83	0,78032	0,98718	16,7424	0,05485	2
0,88	0,78033	0,98718	16,7455	0,05486	2
0,92	0,78034	0,98719	16,7486	0,05487	2
0,96	0,78035	0,98719	16,7517	0,05489	2
1,00	0,78037	0,98719	16,7548	0,05490	2

STORE LEFT 55886 USED 10727

&END;
CPU TIME = 0004 21,387 REAL TIME 00 06 34
A